Section 7.6 Moments, Centers of Mass, and Centroids

Center of Mass in a One-Dimensional System

Let the point masses $m_1, m_2, \ldots, m_n$ be located at $x_1, x_2, \ldots, x_n$.

1. The moment about the origin is $M_0 = m_1 x_1 + m_2 x_2 + \ldots + m_n x_n$
2. The center of mass is $\bar{x} = \frac{M_0}{m}$ where $m$ is the total mass of the system.

Center of Mass in a Two-Dimensional System

Let the point masses $m_1, m_2, \ldots, m_n$ be located at $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$.

1. The moment about the y-axis is $M_y = m_1 x_1 + m_2 x_2 + \ldots + m_n x_n$
2. The moment about the x-axis is $M_x = m_1 y_1 + m_2 y_2 + \ldots + m_n x y_n$.
3. The center of mass (or center of gravity) is $\bar{x} = \frac{M_y}{m}, \bar{y} = \frac{M_x}{m}$ where $m$ is the total mass of the system.
Center of Mass in a Planar Lamina

Let $f$ and $g$ be continuous functions such that $f(x) \geq g(x)$ on $[a, b]$, and consider the planar lamina of uniform density $\rho$ bounded by the graphs of $y = f(x), y = g(x)$, and $a \leq x \leq b$.

The center of mass $(\bar{x}, \bar{y})$ is given by:

$$\bar{x} = \frac{1}{A} \int_{a}^{b} \left[ \frac{f(x) + g(x)}{2} \right] [f(x) - g(x)] \, dx$$

$$\bar{y} = \frac{1}{A} \int_{a}^{b} x[f(x) - g(x)] \, dx$$

where $A$ is the area of the region: $A = \int_{a}^{b} [f(x) - g(x)] \, dx$.

*Note $\bar{x} = \frac{M_x}{m}$. Since $\rho$ will be involved in the calculation of $M_x$ as well as $m$, it will cancel out of the computation of $\bar{x}$.

**Example 1**: Find the center of mass for the laminas of uniform density $\rho$ bounded by the graphs of the equations.

(a) $y = -x + 3, y = 0, x = 0$
(b) \( y = \sqrt{x} + 1, \ y = \frac{1}{3} x + 1 \)
Example 2: The Centroid of a Simple Plane Region

Find the centroid of the region shown.

Find the center of mass of each rectangle. Treat the masses as point masses located at the center of mass of each rectangle, then use the pattern outlined above for finding the center of mass in a two-dimensional system.