Antiderivatives

Given a function f, can we find a function F such that the derivative of F is f.

$f(x) = 2x$ $\qquad F(x) =$

$f(x) = 3x^2$ $\qquad F(x) =$

$f(x) = 5x^3$ $\qquad F(x) =$

$f(x) = 3\sin^2 x \cos x$ $\qquad F(x) =$

$f(x) = \dfrac{x}{\sqrt{1-x^2}}$ $\qquad F(x) =$

$f(x) = \dfrac{1}{\sqrt{1-x^2}}$ $\qquad F(x) =$

$f(x) = \sec^2 x$ $\qquad F(x) =$

$f(x) = \csc 5x \cot 5x$ $\qquad F(x) =$

$f(x) = x^2 - 3x + 2$ $\qquad F(x) =$

A function $F(x)$ is said to be an antiderivative of a function $f(x)$ on an interval I if $F'(x) = f(x)\ \forall\ x \in I$.

Theorem: If F is an antiderivative of f on an interval I, then G is an antiderivative of f on the interval I if and only if G is of the form $G(x) = F(x) + C \; \forall \, x \in I$, where C is a constant. Note: We will make use of the theorem: If $f'(x) = 0 \; \forall \, x \in (a, b)$, then $f(x)$ is constant on $[a, b]$.

$\Rightarrow$ Assume F and G are antiderivatives of f, then prove $G(x) = F(x) + C$

$\Leftarrow$ Assume that $G(x) = F(x) + C \; \forall \, x \in I$, prove that G is an antiderivative of f too.

Recall notation for derivative:

Antiderivative of $3x^2$:

Notation for antiderivative (indefinite integral):

Inverse nature of integration and differentiation

$\int f(x)dx = F(x) + C$ means that F is an antiderivative of f $i.e. F'(x) = f(x)$.

$\int F'(x)dx = \int f(x)dx = F(x) + C$

$$\frac{d}{dx}\left[\int f(x)\,dx\right] = \frac{d}{dx}[F(x) + C] = F'(x) + 0 = f(x)$$

Differentiation formulas

$$\frac{d}{dx}[C] =$$

$$\frac{d}{dx}[kx] =$$

$$\frac{d}{dx}[kf(x)] =$$

$$\frac{d}{dx}[f(x) \pm g(x)] =$$

$$\frac{d}{dx}[x^n] =$$

$$\frac{d}{dx}[\sin x] =$$

$$\frac{d}{dx}[\cos x] =$$

$$\frac{d}{dx}[\tan x] =$$

$$\frac{d}{dx}[\cot x] =$$

$$\frac{d}{dx}[\sec x] =$$

$$\frac{d}{dx}[\csc x] =$$

Integration (Antidifferentiation) formulas

$$\int 0\, dx = C$$

$$\int k\, dx = kx + C$$

$$\int kf(x)\, dx = k\int f(x)dx + C$$

$$\int [f(x) \pm g(x)]\, dx = \int f(x)dx + \int g(x)dx$$

$$\int x^n\, dx = \frac{x^{n+1}}{n+1} + C, \quad n \neq -1$$

$$\int \cos x\, dx = \sin x + C$$

$$\int \sin x\, dx = -\cos x + C$$

$$\int \sec^2 x\, dx = \tan x + C$$

$$\int \csc^2 x\, dx = -\cot x + C$$

$$\int \sec x \tan x\, dx = \sec x + C$$

$$\int \csc x \cot x\, dx = -\csc x + C$$

Examples: Antidifferentiate each of the following.

1. $\int 5dx$

2. $\int (x^2 + 5x + 6)dx$

3. $\int \sqrt{x}\, dx$

4. $\displaystyle\int \frac{1}{\sqrt[3]{x}}\, dx$

5. $\displaystyle\int \frac{x+3}{\sqrt{x}}\, dx$

6. $\displaystyle\int \left(\sqrt[3]{x} + \frac{3}{4\sqrt{x}} \right) dx$

7. $\displaystyle\int \frac{1}{(5x)^2}\, dx$

8. $\int (\sin x + 3\cos x)\, dx$

9. $\displaystyle\int \frac{\cos x}{\sin^2 x}\, dx$

10. $\displaystyle\int \frac{\cos x}{1 - \cos^2 x}\, dx$

A <u>differential equation</u> in x and y is an equation that involves x, y and derivatives of y.

Examples: $y' = 3x$ or $\frac{dy}{dx} = x^2 + 1$

To solve a differential equation means to find a solution (a family of functions) which satisfies the equation.

General solution: Find the family of solutions to the differential equation. This find the general antiderivative of f with a $+C$.

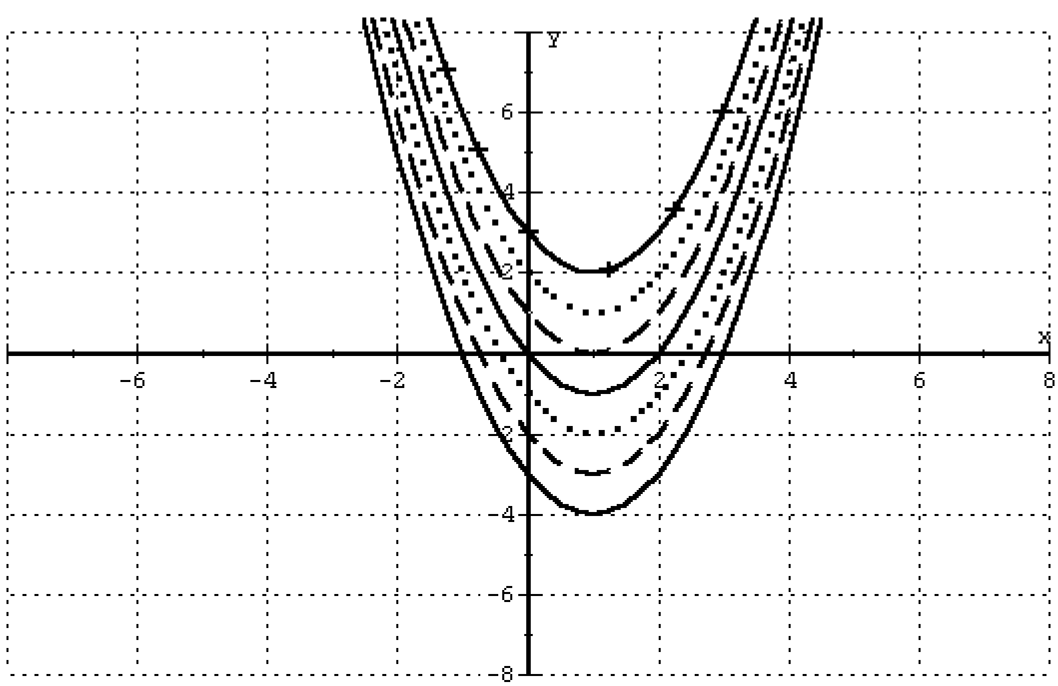
Particular solution: Find the specific solution to the differential equation that passes through the initial condition.

Initial condition: A point that one of the curves passes through that you want to go through.

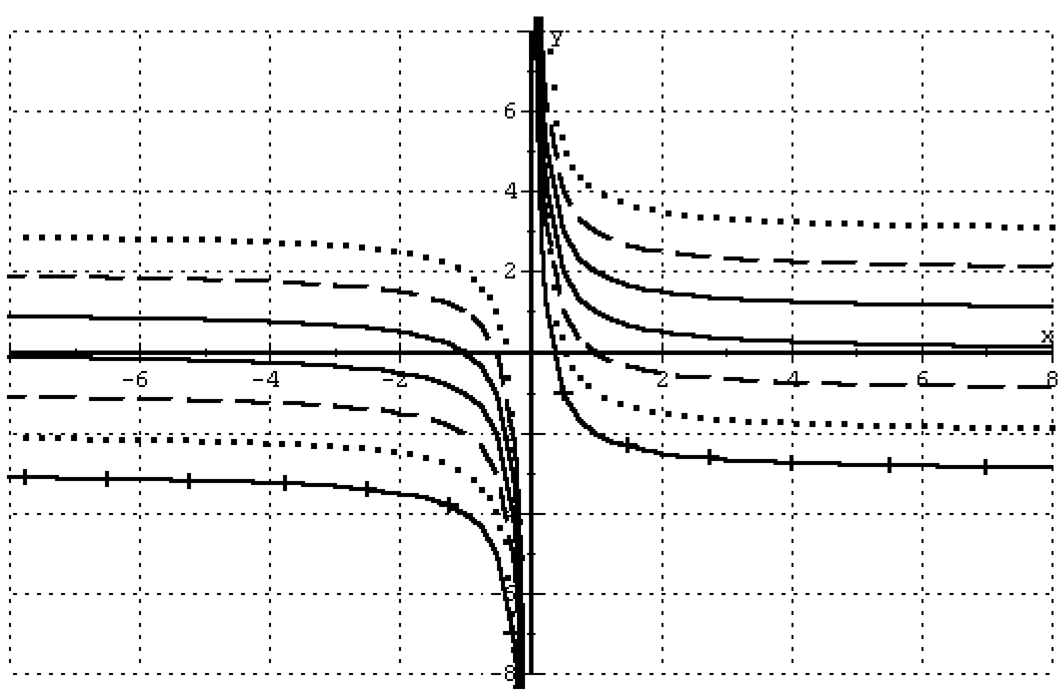
Example: Find the solution to $y' = 3x^2$ with initial condition $(0, 0)$.

Find a general solution to each differential equation and then find a particular solution which satisfies the initial condition.

11. $\dfrac{dy}{dx} = 2(x-1) \;\; f(3) = 2$



12. $y' = -\frac{1}{x^2}$ at $(1, 3)$



13. $f''(x) = x^2,\ \ f'(0) = 6,\ f(0) = 3$

14. The rate of growth $\frac{dP}{dt}$ of a population of bacteria is proportional to the square root of t where P is the population size and t is the time in days $(0 \leq t \leq 10)$. The initial size of the population is 500. After 1 day the population has grown to 600. Estimate the population after 7 days.

$$a = \frac{dv}{dt}$$

$v = \frac{ds}{dt}$

15. A ball is thrown vertically upward with an initial velocity of 40 ft/sec from a point 20 feet above the ground. How long will it take to reach its greatest height? What is its greatest height? How long will it take to reach the point of release on its way down? How long will it take to reach the ground? With what velocity will it reach the ground?

16. A rocket is lifted off the ground with a constant acceleration of 22 meters/second^2. Find the velocity of the rocket and its height above the ground 30 seconds after liftoff.

17. If the brakes of a car can give the car a constant negative acceleration of 8 m/sec^2, what is the greatest speed it may be going if it is necessary to be able to stop the car within 25 meters after the brake is applied?

18. A direction field consists of line segments with slopes given by a differential equation. These lines give a visual perspective of the directions of the solutions of the differential equation. The graph shown below is the direction field for the differential equation $\dfrac{dy}{dx} = x^2 - 1$. Sketch two approximate solutions including the one that passes through the point $(-1, 3)$. Use integration to find the equation of particular solution.