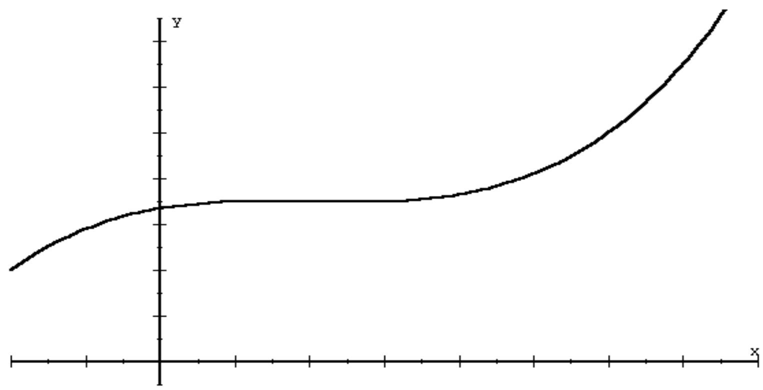
Area Functions

Let $f(x)$ be continuous on $[a, b]$. If $a \leq x \leq b$, then $f(x)$ is continuous on $[a, x]$ and therefore integrable on $[a, x]$.

Consider $\int_a^x f(t)dt$



What is the interpretation of $\int_a^x f(t)dt$?

What happens to $\int_a^x f(t)dt$ when x changes?

$\int_a^x f(t)dt$ is called an

Let $G(x) = \int_a^x f(t)dt$

What is $G(a)$?

1. Let $f(t) = 3, x > 0$

$$G(x) = \int_0^x 3\,dt$$

If $x < 0$

$$G(x) = \int_0^x 3\,dt$$

$$\therefore \quad G(x) = \int_0^x 3\,dt \; = \qquad\qquad \forall\, x$$

2. Let $f(t) = t, x > 0$

$$G(x) = \int_0^x t\,dt$$

3. Let $f(t) = 2 - t, x > 0$

$$G(x) = \int_0^x (2 - t)\,dt$$

4. Let $f(t) = 2 - t$,

$$G(x) = \int_1^x (2 - t)\, dt$$

It appears that an integral with a variable upper limit is an antiderivative of the original function, evaluated at the variable upper limit.

$\int_a^x f(t)dt = G(x)$ where $G'(x) = f(x)$

<u>Intermediate Value Theorem</u>: If $f(x)$ is continuous on a closed interval $[a, b]$ and k is any number between $f(a)$ and $f(b)$, then $\therefore \exists$ at least one number $c \in [a, b]$ such that $f(c) = k$.

<u>Mean Value Theorem for Integrals</u>: If $f(x)$ is continuous on the closed interval $[a, b]$, then $\exists$ a number $c \in [a, b]$ such that $\displaystyle\int_a^b f(x)\, dx = f(c)(b - a)$

Average Value of a Function $= \boldsymbol{\frac{1}{b-a}\int_a^b f(x)\, dx}$

<u>Fundamental Theorem of Calculus</u>: Suppose $f(x)$ is continuous on a closed interval $[a, b]$.

I) If the function $G(x)$ is defined by $G(x) = \displaystyle\int_a^x f(t)dt$ for every x in $[a, b]$, then $G(x)$ is an antiderivative of f on $[a, b]$.

Alternate version of FTC part I: $\boldsymbol{\frac{d}{dx}\int_a^x f(t)\,dt = f(x)}$

II) If F is any antiderivative of f on $[a, b]$, then $\displaystyle\int_a^b \boldsymbol{f(x)dx = F(b) - F(a)}$

Proof:

I) We must show that $G'(x) = f(x)$ i.e. $G'(x) = \lim\limits_{\Delta x \to 0}$

Proof:

II) If $F(x)$ is any antiderivative of $f(x)$ on $[a, b]$, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

We know that $G(x) = \int_a^x f(t)dt$ is an antiderivative of $f(x)$. If we also assume that F is an antiderivative of f we know that $G(x) = F(x) + c$

$$\Rightarrow \int_a^x f(t)dt = F(x) + c \quad \forall\, x \in [a, b]$$

$$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a)$$

Examples: Evaluate each of the following definite integrals using the FTC

5. $\displaystyle\int_1^4 7\,dx$

6. $\displaystyle\int_0^3 \big(3x^2 + x - 2\big)\,dx$

7. $\displaystyle\int_{-2}^{-1} \left(u - \frac{1}{u^2}\right) du$

8. $\displaystyle\int_0^2 (2 - t)\sqrt{t}\,dt$

9. $\displaystyle\int_0^4 \left|x^2 - 4x + 3\right| dx$

10. $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (2t + \cos t)\, dt$

11. Determine the area of the region having the given boundaries
$y = -x^2 + 3x \qquad y = 0$

12. Find the value of c guaranteed by the Mean Value Theorem for Integrals if $f(x) = \dfrac{9}{x^3}$ on $[1, 3]$.

13. Use the FTC to find $F'(x)$ if $F(x) = \displaystyle\int_1^x \frac{t^2}{t^2+1}\,dt$

14. Find $F'(x)$ for the following:

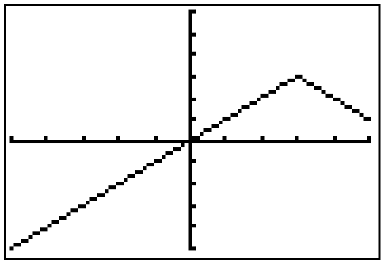
a) $F(x) = \int_0^x t\left(t^2+1\right) dt$

b) $F(x) = \int_4^{sinx} \sqrt{t}\, dt$

c) $F(x) = \int_{x^2}^{3} \sqrt[3]{t^2+1}\, dt$

d) $F(x) = \int_{-x}^{2x} \cos(\sqrt{t})\, dt$

15. Find $\int_1^3 f'(x)\,dx$ if f has the graph below:



16. Let $g(x) = \int_0^x f(t)\,dt$, where f is the function whose graph is shown below.

a) Estimate $g(0),\ g(2),\ g(4),\ g(6)$ and $g(8)$.

b) Find where g is increasing and decreasing.

c) Find where g is concave up and concave down.

d) Find any extrema of g.

17. The acceleration for a particle moving in a straight line is given by $a(t) = 2t - 1$. The initial velocity is $v(0) = -2\,m/s$.

a) Find $v(t)$

b) Find the net change in position from $t = 0$ to $t = 3$ seconds.

c) Find the total distance traveled from $t = 0$ to $t = 3$ seconds.

18. The following Riemann Sum represents an area under a curve:

$$\lim_{n\to\infty}\sum_{i=1}^{n}(1+\tfrac{2i}{n})^2(\tfrac{2}{n})$$

Identify the definite integral that this represents.

Draw the area this represents.

Find the area this represents.