Basic Differentiation Rules and Rates of Change:

1. Derivative of a constant function $f(x) = c$

Proof: $$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$\boldsymbol{\frac{d}{dx}(c) = 0}$$

2. Derivative of a function of the form $f(x) = x^n$ where n is a rational number

$\boldsymbol{\frac{d}{dx}x^n = nx^{n-1}}$ where n is a positive integer.

Note: To be differentiable at 0, x^{n-1} must be defined on an interval containing 0

3. Derivative of a constant times a function.

$$\frac{d}{dx}cf(x) = cf'(x)$$

4. Derivative of the sum/difference of two functions
Assuming $f(x)$ and $g(x)$ are differentiable

$$\frac{d}{dx}[(f \pm g)(x)] = \frac{d}{dx}[f(x) \pm g(x)] = f'(x) \pm g'(x)$$

5. Derivative of $f(x) = \sin x$

Proof: $f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$

$$\frac{d}{dx}\sin x = \cos x$$

6. Derivative of $f(x) = \cos x$

$$\frac{d}{dx}\cos x = -\sin x$$

7. Derivative of $f(x) = e^x$

Proof: $$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

$$\boldsymbol{\frac{d}{dx} e^x = e^x}$$

Examples: Find the derivative of each of the following:

1. $f(x) = 2$

2. $g(x) = 3x - 1$

3. $f(x) = 2x^3 - x^2 + 3x$

4. $y = 5x^2 + 2\sin x + e^x$

5. $f(x) = 3\cos x - \sqrt{x} - e^x$

6. $f(x) = \dfrac{2}{3x^2}$

7. $y = \dfrac{\pi}{(7x)^2}$

8. $y = \dfrac{2x^2 - 3x + 1}{x}$

9. $y = \sqrt[3]{x} + \sqrt[5]{x^3}$

10. Find the derivative of $f(x) = 3 - \dfrac{3}{5x}$ at $\left(\dfrac{3}{5}, 2\right)$

11. Find the equation of the tangent line to $f(x) = 3(5-x)^2$ at $(5, 0)$

12. Find the equation of the tangent line to $f(t) = 2 + 3\cos t$ at $(\pi, -1)$.

13. Determine points where the tangent line is horizontal.

$f(x) = x^3 + x$ $\qquad$ $g(x) = \sqrt{3}x + 2\cos x,\ 0 \le x \le 2\pi$

14. Find the equation of the tangent line to $f(x) = \dfrac{2}{x}$ through the point $(5, 0)$.
Note: $(5, 0)$ is not on the graph.

15. Find the average rate of change for $f(t) = t^2 - 3$ on the interval $[2, 2.1]$.

Find the instantaneous rate of change for the above function at $t = 2$.

A common use for rate of change is to describe the motion of an object moving in a straight line.

Positive Direction: Negative Direction:

Typically $s(t)$ represents the position function of the object. Therefore,

Average Rate of Change ------> **Average Velocity**

Instantaneous Rate of Change------> **Instantaneous Velocity**

The position of a free-falling object (neglecting air resistance) under the influence of gravity can be represented by the equation:

$$\boldsymbol{s(t) = \frac{1}{2}gt^2 + v_0 t + s_0}$$

where s_0 is the initial height of the object, v_0 is the initial velocity of the object, and g is the acceleration due to gravity. On Earth, the value of g is approximately -32 feet per second^2 or -9.8 meters per second^2.

16. Let $s(t) = -16t^2 + 32t + 48$ represent the height of an object propelled from a cliff 48 feet high.

a) Find the average velocity over the following time intervals:

$[1, 2]$ $[2, 2.1]$

b) Find the (instantaneous) velocity when $t = 2$ seconds:

c) Find the time when the object reaches its maximum height:

d) Find the maximum height obtained by the object:

17. At time $t = 0$, a diver jumps from a platform diving board that is 32 feet above the water. The position of the diver is given by

$$s(t) = -16t^2 + 16t + 32$$

where s is measured in feet and t is measured in seconds.

a) When does the diver hit the water?

b) What is the diver's velocity at impact?

18. The graph of the position function is given. Sketch the graph of the velocity function.

19. The graph of the velocity function is given. Sketch a graph of the position function.

20. The volume of a cube with sides of length s is $V = s^3$. Find the rate of change of the volume with respect to s when $s = 4$ cm.