Concavity and the Second Derivative Test

The graph of f lies above its tangent line
f' is increasing so:

The graph of f lies above its tangent line
f' is decreasing so:

Definition:
Suppose that f is a function whose derivative f' exists at every point in an open interval I, then

1. f is **concave up** on the interval I, if ____________________ over I.

2. f is **concave down** on the interval I, if __________________ over I.

Test for Concavity:
1. If $f''(x) > 0$ over an interval I, then f is concave up on I. $(turns\,up)$
2. If $f''(x) < 0$ over an interval I, then f is concave down on I. $(turns\,down)$

Four Possible Scenarios - piecing together these four shapes gives the graph of a function

If a graph is concave up, will it have a relative max or relative min?

If a graph is concave down, will it have a relative max or relative min?

Second-Derivative Test:
Suppose that f is a function for which $f'(x)$ exists for all x in an open interval (a,b) contained in its domain, and that there is a critical point c in (a,b) for which $f'(c)=0,$ then,

1. $f(c)$ is a relative minimum if $f''(c)>0$
2. $f(c)$ is a relative maximum if $f''(c)<0$

The test fails if $f''(c)=0.$ Use the First Derivative test in this case.
Note: If $f'(c)$ is DNE, then $f''(c)$ is DNE, so you can't use the second derivative test.

Definition: A point $(c, f(c))$ on the graph of $f(x)$ is a **point of inflection** if the following conditions are satisfied:

i) $f(x)$ is continuous at $c.$
ii) There is an open interval (a,b) containing c such that the graph is concave upward on (a,c) and concave downward on (c,b) or vice versa.
iii) There is a tangent line to the curve at $(c, f(c)).$

If $f''(x)$ changes sign at a number c and if there is a tangent line at $(c, f(c))$, then $(c, f(c))$ is a point of inflection.

Possible points of inflection (PPI) will occur where $f''(x)=0$ or where $f''(x)$ DNE.

To test a PPI you must check for a change in concavity AND the existence of a tangent line at that point.

Strategy for Curve Sketching

1. **Find f' and f''**

2. **Find critical points of f' ($f' = 0$ or DNE)** | **Find critical points of f'' ($f'' = 0$ or DNE)**

3. **Make a number line for f'** | **Make a number line for f''**

4. **Check test values in f' to determine $+ / -$** | **Check test values in f'' to determine $+ / -$**

5. **Line number lines on top of each other:**

   ______________________ $f'(x)$

   ______________________ $f''(x)$

6. **Find y-values of all important points using original function f and PLOT relative extrema, POI, intercepts, and any other necessary points**

7. **Using four possible scenarios, piece together the appropriate shapes of the graph**

Find intervals where the function is increasing, decreasing, concave up, concave down, coordinates of relative extrema, points of inflection, and sketch a graph.

1. $f(x) = 2x^3 - 3x^2 - 12x + 8$

$f'(x) =$

$f''(x) =$

Increasing:

Decreasing:

Concave up:

Concave down:

Relative min:

Relative max:

Pt(s) of inflection:

2. $f(x) = 5x^{\frac{2}{3}} - x^{\frac{5}{3}}$

$$f'(x) = \frac{5(2-x)}{3x^{\frac{1}{3}}}$$

$$f''(x) = \frac{-10(1+x)}{9x^{\frac{4}{3}}}$$

Increasing:

Decreasing:

Concave up:

Concave down:

Relative min:

Relative max:

Pt(s) of inflection:

3. Use the Second Derivative Test to find relative extrema:

$f(x) = 2x^3 + 3x^2 - 12x + 8$

$f'(x) =$

$f''(x) =$

Contrast with First Derivative Test

$f(x) = xe^{-x}$

$f'(x) =$

$f''(x) =$

$f(x) = 2\sin x + \cos 2x$ on $[0, 2\pi]$

$f'(x) =$

$f''(x) =$

4. Sketch the graph of a function $f(x)$ having the given characteristics: $f(0) = f(2) = 0,\ f'(x) > 0$ if $x < 1,\ f'(1) = 0,$ $f'(x) < 0$ if $x > 1,\ f''(x) < 0$

5. Sketch the graph of $f(x) = \dfrac{x}{x^2+1}$

$$f'(x) = \frac{(1-x)(1+x)}{(x^2+1)^2}$$

$$f''(x) = \frac{-2x(3-x^2)}{(x^2+1)^3}$$

6. Sketch the graph of $f(x) = x\sqrt{4-x^2}$

$$f'(x) = \frac{4-2x^2}{\sqrt{4-x^2}}$$

$$f''(x) = \frac{2x(x^2-6)}{(4-x^2)^{\frac{3}{2}}}$$

7. $f(x) = 2x^{\frac{1}{3}} - x^{\frac{2}{3}}$

$$f'(x) = \frac{2\left(1 - x^{\frac{1}{3}}\right)}{3x^{\frac{2}{3}}}$$

$$f''(x) = \frac{2\left(x^{\frac{1}{3}} - 2\right)}{9x^{\frac{5}{3}}}$$

8. Sketch the graph of $f(x) = 2x + \cot x$ on $(0, \pi)$

$f'(x) = 2 - \csc^2 x$

$f''(x) = 2\csc^2 x \cot x$