Evaluating Limits Algebraically with Limit Laws

There are some general situations in which $\lim_{x \to c} f(x) = f(c)$, in which case, limits can be evaluated by **direct substitution**.

Consider: $f(x) = b$ $\qquad$ $f(x) = x$ $\qquad$ $f(x) = x^n$

Theorem: **Basic Limits**

Let b and c be real numbers and let n be a positive integer. Then,

1. $\lim_{x \to c} b =$ $\qquad$ 2. $\lim_{x \to c} x =$ $\qquad$ 3. $\lim_{x \to c} x^n =$

$\lim_{x \to 2} 4 =$ $\qquad$ $\lim_{x \to 2} x =$ $\qquad$ $\lim_{x \to 2} x^2 =$

Theorem: **Properties of Limits**

Let b and c be real numbers, let n be a positive integer, and let f and g be functions with the following limits.

$\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = K$

1. Scalar multiple: $\lim_{x \to c} [bf(x)] = b \lim_{x \to c} f(x) =$

2. Sum or Difference: $\lim_{x \to c} [f(x) \pm g(x)] = \lim_{x \to c} f(x) \pm \lim_{x \to c} g(x) =$

3. Product: $\lim_{x \to c} [f(x) \cdot g(x)] = \lim_{x \to c} f(x) \cdot \lim_{x \to c} g(x) =$

4. Quotient: $\lim_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)} =$

5. Power: $\lim_{x \to c} [f(x)]^n = [\lim_{x \to c} f(x)]^n =$

$\lim_{x \to 3} (2x^3 - 3x + 1) =$

Note: $\lim_{x \to 3} (2x^3 - 3x + 1) =$

Theorem: **Limits of Polynomial and Rational Functions**

Direct substitution is valid for finding limits of all **polynomial functions**. That is, if p is a polynomial function and c is a real number, then $\boldsymbol{\lim_{x \to c} p(x) =}$

Also, ***direct substitution*** is valid for finding limits of all **rational functions**, given by $r(x) = \dfrac{p(x)}{q(x)}$, provided that c is a real number such that $q(c) \neq 0$. That is, $\boldsymbol{\lim_{x \to c} r(x) = \lim_{x \to c} \dfrac{p(x)}{q(x)} =}$, $q(c) \neq 0$.

$$\lim_{x \to 1} \frac{x^2 - x - 2}{x^3 + 2} =$$

Theorem: Limit of Functions Involving a Radical

Direct substitution is valid for finding limits of all **radical functions,** given that the radical is defined at c. That is,
$\boldsymbol{\lim_{x \to c} \sqrt[n]{x} =}$, for all c if n is odd, and for $c > 0$ if n is even.

$$\lim_{x \to -27} \sqrt[3]{x} =$$

$$\lim_{x \to 16} \sqrt[4]{x} =$$

Theorem: The Limit of a Composite Function

If f and g are functions such that $\lim_{x \to c} g(x) = L$ and $\lim_{x \to L} f(x) = f(L)$, then $\lim_{x \to c} f(g(x)) =$

$$\lim_{x \to 3} \sqrt{2x^2 - 3x} =$$

Theorem: **Limits of Trigonometric Functions**

Let c be a real number in the domain of the given trigonometric function. In each case ***direct substitution*** is valid for finding the limit of the given **trigonometric function.**

1. $\lim_{x \to c} \sin x =$
2. $\lim_{x \to c} \cos x =$
3. $\lim_{x \to c} \tan x =$
4. $\lim_{x \to c} \cot x =$
5. $\lim_{x \to c} \sec x =$
6. $\lim_{x \to c} \csc x =$

$\lim_{x \to \frac{\pi}{2}} \sin x =$

$\lim_{x \to \frac{\pi}{6}} \cos x =$

$\lim_{x \to \frac{\pi}{4}} \tan x =$

Given $\lim_{x \to c} f(x) = 2$ and $\lim_{x \to c} g(x) = 4$, find each of the following:

$\lim_{x \to c} [4f(x)]$

$\lim_{x \to c} (f(x) + g(x))$

$\lim_{x \to c} \frac{f(x)}{4}$

$\lim_{x \to c} \sqrt[3]{f(x)}$

$\lim_{x \to c} (f(x))^2$

We will now examine limits that **cannot** be evaluated using ***direct substitution.***

Consider $f(x) = \dfrac{x^2 - 3x}{x}$. Our limit theorem for rational functions is not valid if we want to find $\lim\limits_{x \to 0} \dfrac{x^2 - 3x}{x}$ since $\lim\limits_{x \to 0} \dfrac{x^2 - 3x}{x} \to$ if we use direct substitution.

Note, however, that $f(x) = \dfrac{x^2 - 3x}{x} =$

Let $g(x) =$ Then $f(x)$ and $g(x)$

Graph of f :

Graph of g :

Theorem: Functions That Agree at All But One Point

Let c be a real number and let $f(x) = g(x)$ for all $x \neq c$ in an open interval containing c. If the limit of $g(x)$ as x approaches c exists, then the limit of $f(x)$ also exists and $\lim\limits_{x \to c} f(x) = \lim\limits_{x \to c} g(x)$.

$$\lim_{x\to 0} \frac{x^2 - 3x}{x} =$$

$$\lim_{x\to 0} \frac{x^3 + 1}{x + 1}$$

$$\lim_{x\to 0} \frac{2 - x}{x^2 - 4}$$

$$\lim_{x\to 0} \frac{\sqrt{2 + x} - \sqrt{2}}{x}$$

$$\lim_{x\to 2} \frac{2 - x}{x^2 - 4}$$

$$\lim_{x\to 0} \frac{\dfrac{1}{x + 4} - \dfrac{1}{4}}{x}$$

$$\lim_{x\to 0} \frac{\sqrt{x + 1} - 2}{x - 3}$$

$$\lim_{h \to 0} \frac{(x+h)^3 - x^3}{h}$$

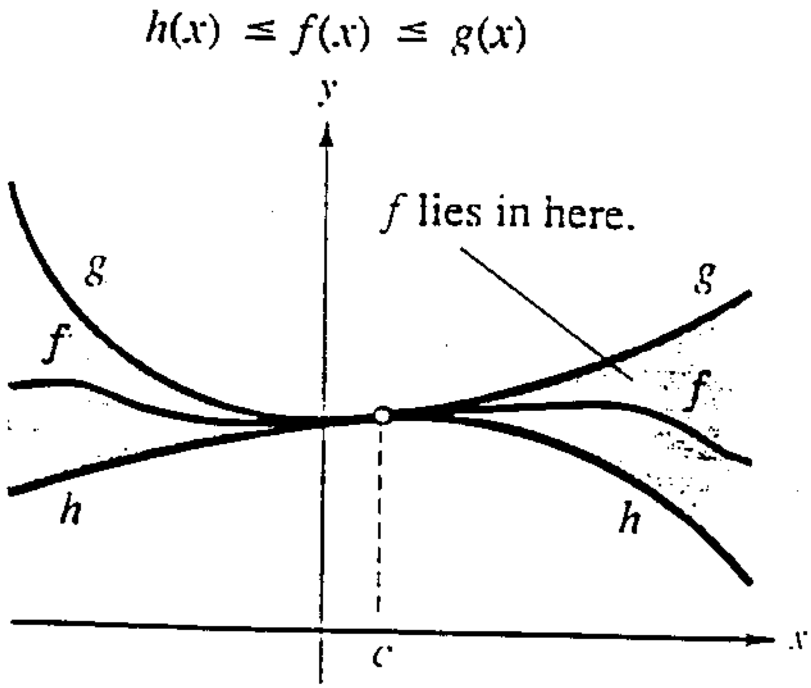
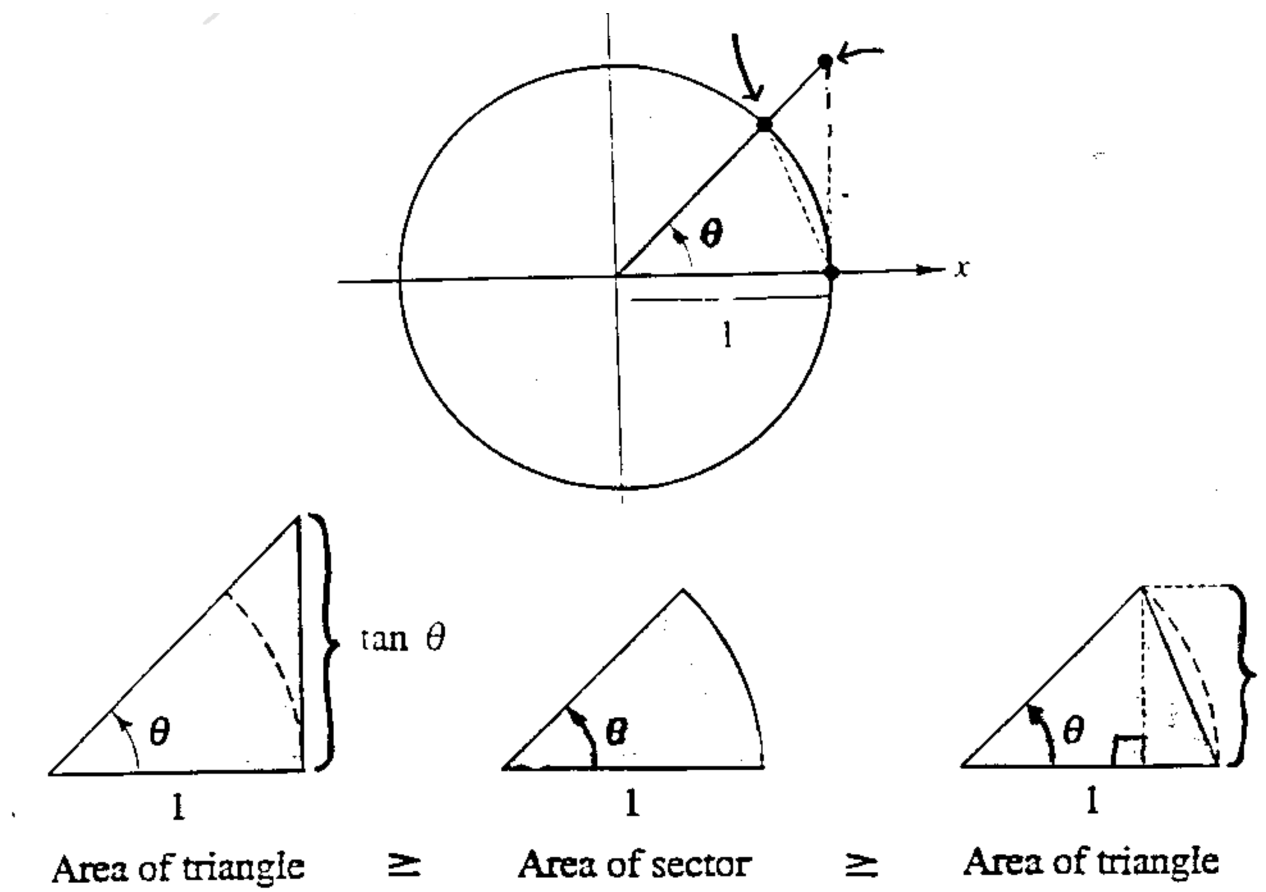


FIGURE 1.19
The Squeeze Theorem

Squeeze Theorem: If $h(x) \le f(x) \le g(x)$ and $\lim_{x \to c} h(x) = L = \lim_{x \to c} g(x)$,
then $\lim_{x \to c} f(x) = L$

Prove: $\lim_{x \to 0} \frac{\sin x}{x}$



Find $\lim_{x \to 0} \dfrac{1 - \cos x}{x}$

<u>Other examples</u>

(might need to make use of $\lim_{x \to 0} \dfrac{\sin x}{x} = 1$ and $\lim_{x \to 0} \dfrac{1 - \cos x}{x}$)

$\lim_{x \to 0} \dfrac{\sin x}{3x}$

$\lim_{x \to 0} \dfrac{\sin 3x}{x}$

$\lim_{x \to 0} \dfrac{\sin 2x}{\sin 3x}$

$\lim_{\theta \to 0} \dfrac{\tan \theta \cos \theta}{\theta}$

$$\lim_{x \to \frac{\pi}{4}} \frac{1 - \tan x}{\sin x - \cos x}$$

$$\lim_{x \to 0} \frac{3\cos x - 2}{\tan x + 1}$$

$$\lim_{x \to 0} x \cos \frac{1}{x}$$

Find $\lim_{x \to a} f(x)$ given that $b - |x - a| \le f(x) \le b + |x - a|$

Limit Theorem Summary

Limits that can be evaluated by direct substitution. The limit properties $(5-9)$ which allow us to simplify expressions so that the limits may be evaluated by direct substitution require that the $\lim_{x \to a} f(x)$ & $\lim_{x \to a} g(x)$ both exist.

1. $\lim_{x \to a}(mx+b) = ma+b$

2. $\lim_{x \to a} C = C$

3. $\lim_{x \to a} x = a$

4. $\lim_{x \to a} x^n = a^n$

5. $\lim_{x \to a} bf(x) = b \lim_{x \to a} f(x)$

6. $\lim_{x \to a}(f(x) \pm g(x)) = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$

7. $\lim_{x \to a}(f(x) \bullet g(x)) = \lim_{x \to a} f(x) \bullet \lim_{x \to a} g(x)$

8. $\lim_{x \to a}\left(\frac{f(x)}{g(x)}\right) = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$ assuming $\lim_{x \to a} g(x) \neq 0$

9. $\lim_{x \to a}[f(x)]^n = \left[\lim_{x \to a} f(x)\right]^n$

10. $\lim_{x \to a} P(x) = P(a)$ where $P(x)$ is a polynomial function

11. $\lim_{x \to a}\left(\frac{p(x)}{q(x)}\right) = \frac{p(a)}{q(a)}$ where $p(x)$ & $q(x)$ are polynomial functions assuming $q(a) \neq 0$

12. $\lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a}$ Assuming $a \in D\left(\sqrt[n]{x}\right)$

13. $\lim_{x \to a}(f \circ g)(x) = (f \circ g)(a)$ Assuming $f(x)$ is continuous at $g(a)$

14. $\lim_{x \to a} \text{Trig}\,(x) = \text{Trig}\,(a)$ Assuming $a \in D\,(\text{Trig}\,(x))$

We now have three methods for finding limits:

1. **Numerically - Make a table of values as x approaches a**
2. **Graphically - Look at a graph of the function near $x = a$**
3. **Algebraically - Plug-in a or use algebra to simplify first and then plug-in a**