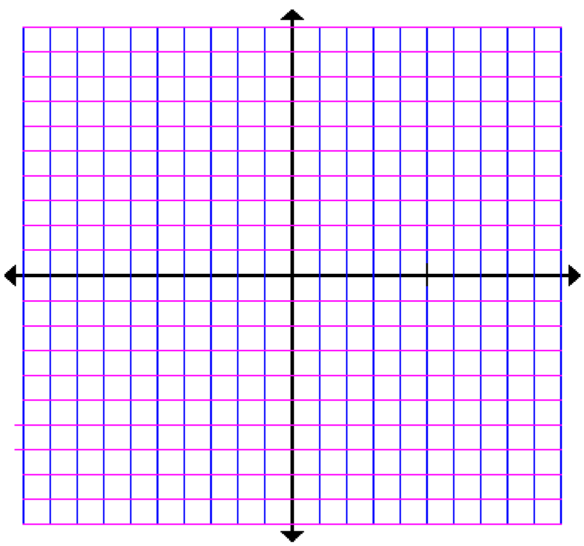
Finding Limits Graphically and Numerically

One fundamental concept used in Calculus is the idea of a limit. We will learn an intuitive approach to the limit today.

Example: Let $f(x) = x^2 - x + 2$. Examine the behavior of $f(x)$ near $x = 2$.

x	$f(x)$
1.0	
1.5	
1.9	
1.95	
1.99	
1.995	
1.999	
1.9999	
2	

x	$f(x)$
3.0	
2.5	
2.1	
2.05	
2.01	
2.005	
2.001	
2.0001	
2	



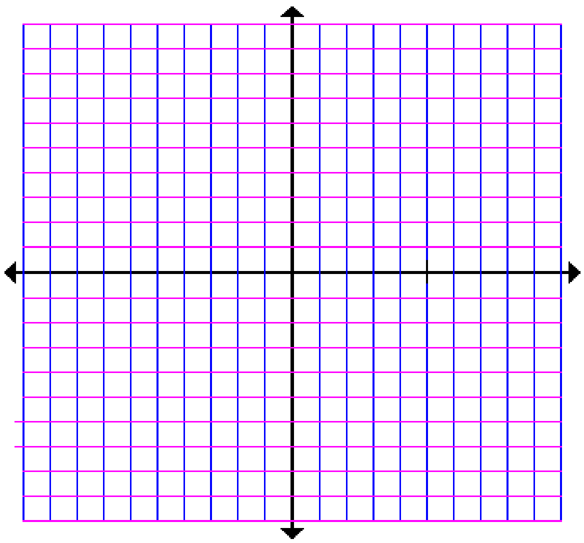
Notation: $\lim_{x\to 2} f(x) = \lim_{x\to 2} x^2 - x + 2 =$

Example: Let $g(x) = \frac{x^2-9}{x-3}$ near $x = 3$

Notice: $\frac{x^2-9}{x-3} =$

x	$f(x)$
2.1	
2.9	
2.99	
2.999	

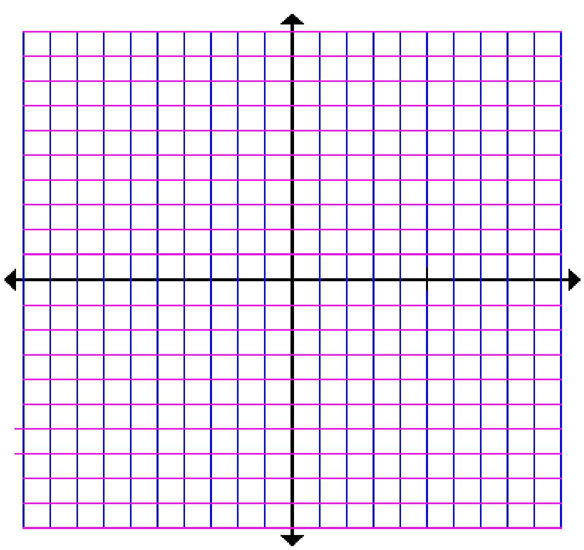
x	$f(x)$
3.1	
3.01	
3.001	
3.0001	



$\lim_{x\to 3} \frac{x^2-9}{x-3} =$

Example: $h(x) = \begin{cases} x+1 & \text{if } x \neq 1 \\ 5 & \text{if } x = 1 \end{cases}$

Find $\lim_{x\to 1} h(x)$

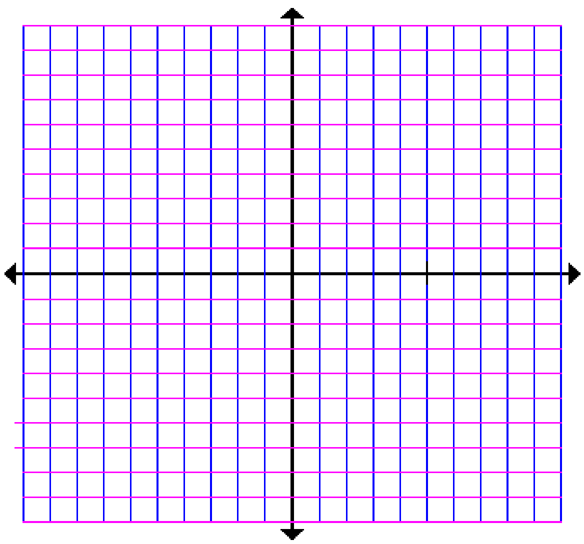


Example: $f(x) = \frac{sinx}{x}$ Find $\lim_{x \to 0} f(x)$

x	$f(x)$
.1	
.01	
.001	
.0001	
.00001	

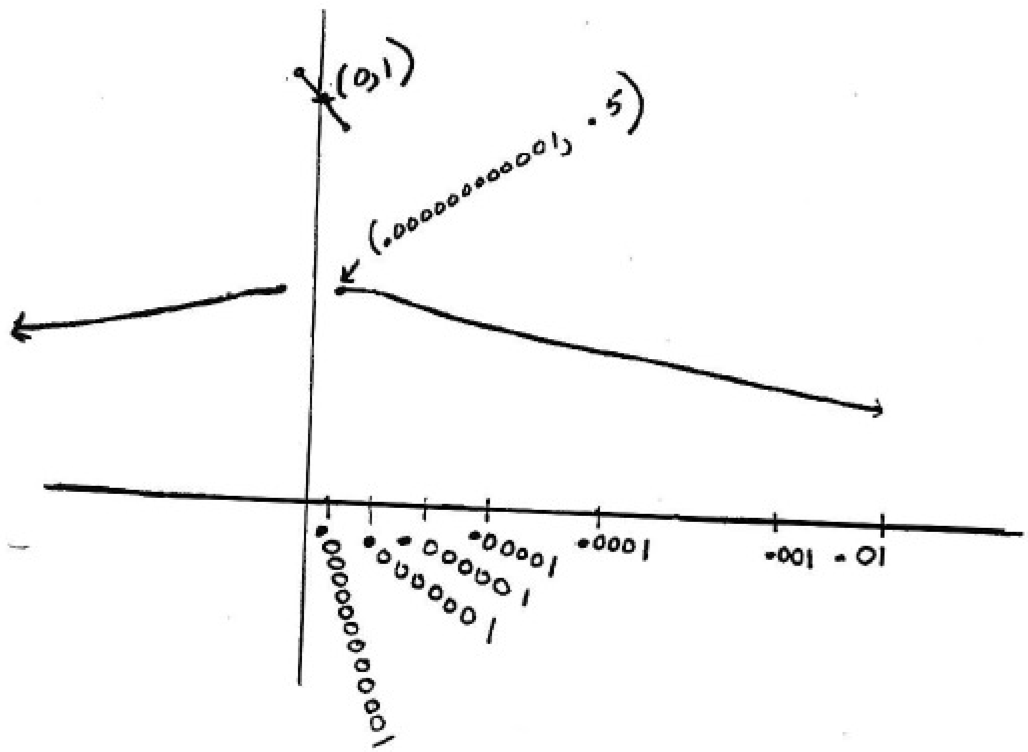
x	$f(x)$
$-.1$	
$-.01$	
$-.001$	
$-.0001$	
$-.00001$	

Example: Estimate $\lim_{x \to 0} sin\frac{1}{x}$ graphically.



Example: Estimate $\lim_{x \to 1} \frac{lnx}{tan\pi x}$ numerically.

Limitations of numerical and graphical approaches for determining a limit:

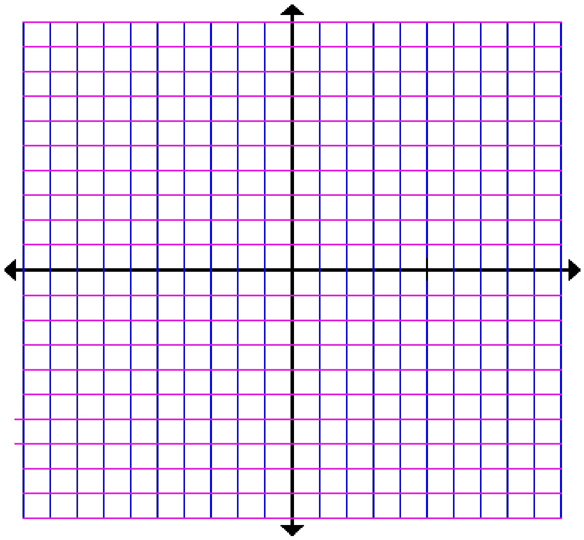


A numerical approach would most likely yield an estimate of the limit $\lim_{x \to 0} f(x) =$

But, in fact, $\lim_{x \to 0} f(x) =$

Heavyside Function:

Example: $j(x) = \frac{|x|}{x}$ Find $\lim_{x \to 0} j(x)$.

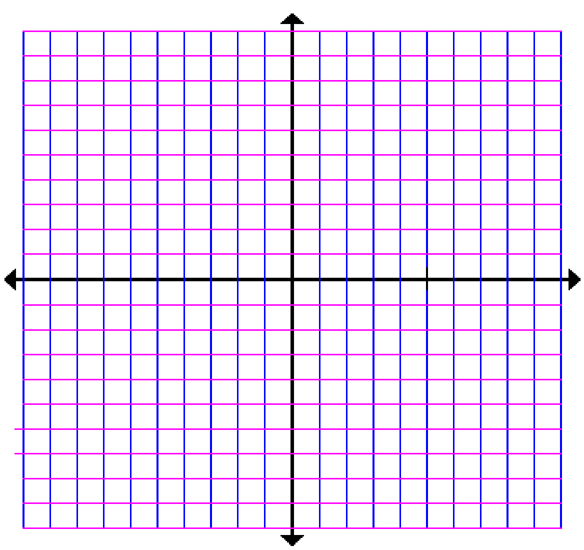


One-Sided Limits:

Right-Hand Limit: $\lim_{x \to a^+} f(x) = L$ $\quad$ **$x \to a^+$ means that x approaches a for values greater than a.**

Left-Hand Limit: $\lim_{x \to a^-} f(x) = L$ $\quad$ **$x \to a^-$ means that x approaches a for values less than a.**

Example: $p(x) = \frac{1}{x^3}$ $\lim_{x \to 0} p(x)$.



The general idea of a limit: Suppose that f is a function defined **near, but not necessarily at, $x = a$**

To say $\lim_{x \to a} f(x) = L$ means informally, that $f(x)$ approaches L as x approaches a.

In other symbols,

Limits matter in Calculus because there are often lots of functions in Calculus that have gaps (holes, asymptotes) in their domains. This often happens when you are trying to find a derivative.
Most limits encountered in Calculus are relatively simple and straightforwad. You can solve them algebraically (plug in), graphically, or numerically (with a table of values).

Informal Definition of a Limit: If $f(x)$ tends to a single number L as x approaches a, then $\lim_{x \to a} f(x) = L$.

Formal Definition of a Limit: Suppose that for every positive number ϵ, no matter how small, there is a positive number δ, so that $|f(x) - L| < \epsilon$ whenever $0 < |x - a| < \delta$.

Comments:

1) Can estimate limits becasue if $\lim_{x \to a} f(x) = L$, then $f(x) \approx L$ if $x \approx a$.
   So we can use tables or graphs near $x = a$.

2) A gap at a? No problem because $\lim_{x \to a} f(x)$ may exist even if $f(a)$ is undefined.
   (Usually $f(a)$ is undefined in the interesting cases.)

3) Even if $f(a)$ is defined, there is no guarantee that $\lim_{x \to a} f(x) = f(a)$.
   The limit definition doesn't look at $f(a)$. It only looks at x's near a ($x \neq a$).

One-Sided Limits:

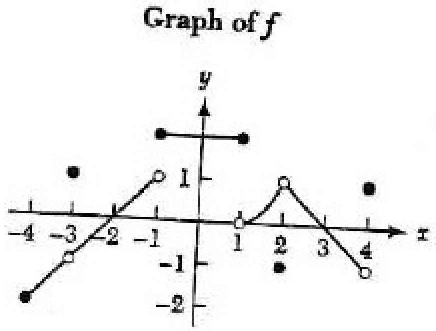
Right-Hand Limit: $\lim_{x \to a^+} f(x) = L$ means that $f(x) \to L$ as $x \to a$ from the right ($x > a$)

Left-Hand Limit: $\lim_{x \to a^-} f(x) = L$ means that $f(x) \to L$ as $x \to a$ from the left ($x < a$)

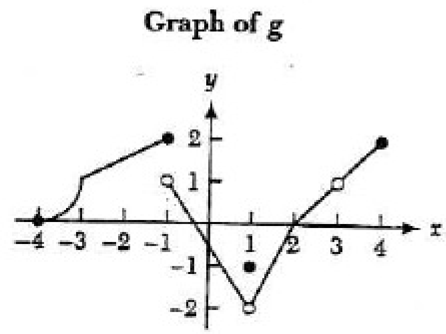
Theorem: The ordinary limit $\lim_{x \to a} f(x)$ exists if and only if both corresponding one-sided limits exist and are equal.

$$\lim_{x \to a} f(x) = L \Leftrightarrow \lim_{x \to a^+} f(x) = L \Leftrightarrow \lim_{x \to a^-} f(x) = L$$

Let $f(x)$ be the graph below:



Let $g(x)$ be the graph below:



Find the following limits:

$\lim_{x\to 0} f(x)$

$\lim_{x\to 2^+} f(x)$

$\lim_{x\to -3} f(x)$

$\lim_{x\to 2} f(x)$

$\lim_{x\to -1} f(x)$

$\lim_{x\to -2^-} f(x)$

$\lim_{x\to 2^-} f(x)$

$\lim_{x\to 1^+} f(x)$

$\lim_{x\to -4^+} f(x)$

$\lim_{x\to -1^+} f(x)$

$\lim_{x\to 4^-} f(x)$

$\lim_{x\to -1^-} f(x)$

$\lim_{x\to -4^+} g(x)$

$\lim_{x\to 0^-} g(x)$

$\lim_{x\to -3} g(x)$

$\lim_{x\to 1} g(x)$

$\lim_{x\to -2} g(x)$

$\lim_{x\to 2} g(x)$

$\lim_{x\to -1^+} g(x)$

$\lim_{x\to 3} g(x)$

$\lim_{x\to -1} g(x)$

$\lim_{x\to 4^-} g(x)$

Recall the formal definition of the limit:

<u>Definition</u>: Let f be a function that is defined on an open interval containing $a,$ except possibly at a itself. Then $\lim_{x \to a} f(x) = L$ if for any $\epsilon > 0$ (no matter how small), there exists a $\delta > 0$ such that if

$$a - \delta < x < a + \delta, \text{ then } L - \epsilon < f(x) < L + \epsilon.;$$
$$(|x - a| < \delta) \qquad\qquad |f(x) - L| < \epsilon$$

This is called the ϵ-δ definition of the limit.

The idea behind this is that you pick any small ϵ (so that you can get as close to L as you want), then you will have to be within δ of a. The size of δ is dependent on the size of ϵ. Usually we find this dependency relationship to help prove our limit-definition.

Example: Given the limit $\lim_{x\to 1}(4x-3)=1$ find δ such that $\epsilon=.01$.
(We are given $\epsilon=.01$ and we need to find a δ that works.)

Example: Use the ϵ-δ definition of a limit to prove that $\lim_{x\to 3}(2x-5)=1$.
We must show that for every $\epsilon>0$, there exists a $\delta>0$ such that $|f(x)-L|<\epsilon$ whenever $|x-a|<\delta$.

Example: What if we mistakenly said $\lim_{x\to 1}(2x+3)=4$?
Let $\epsilon=\frac{1}{2}$. Can we find a δ such that if $1-\delta<x<1+\delta$, then $4-\frac{1}{2}<2x+3<4+\frac{1}{2}$?

Example: Use the ϵ-δ definition of a limit to prove that $\lim_{x \to 3} x^2 = 9$.

We must show that for every $\epsilon > 0$, there exists a $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $|x - a| < \delta$.

Example: Given $\lim_{x \to 8} \sqrt[3]{x} = 2$ and $\epsilon = .01$, find the maximum value of δ, accurate to 6 decimal places.

Example: Assume that $\lim_{x\to 0} f(x)$ exists for the following. For any $\epsilon > 0$ discuss why it is impossible to find a value for δ such that if $0 < |x - a| < \delta$, then $|f(x) - L| < \epsilon$, hence contradicting your assumption. Use a rough sketch to support your argument.

$$\lim_{x\to 0} \frac{1}{x^2} = 0$$