Linear & Quadratic Approximations Lab

Purpose: To learn about polynomial approximations to a function near some point on the graph of the function. We will place particular emphasis on linear (degree 1) and quadratic (degree 2) approximations to $f(x)$ near a point $(a, f(a))$.

Introduction: As we have seen, polynomials are "nice" functions in that they are continuous and differentiable. Calculations of function values of a polynomial just arithmetic operations (addition, subtraction, multiplication and division). So if we have a polynomial $P(x)$, simple arithmetic will allow us to calculate, say, $P(1.3)$ (contrast this with the calculation of $\sqrt[3]{1.3}$). While we could use a calculator to find these values, do we understand what the calculator is doing? How would we get these values "by hand"?

Exercise 1: Tangent Line and Tangent Parabola

For this exercise, consider the function $y = f(x) = \sqrt[3]{x}$. Use the first and second derivative to find the tangent line and tangent parabola to $f(x)$ at $x = 1$,

a) **Tangent Line:** $\boldsymbol{l(x) = f'(a)(x-a) + f(a)}$

b) **Tangent Parabola:** $\boldsymbol{q(x) = \frac{f''(a)}{2}(x-a)^2 + f'(a)(x-a) + f(a)}$

c) Graph the function and its tangent line and tangent parabola together in a graphing window centered at the point $(1, f(1))$ (you may use Derive or Maple), and answer the following questions: (include a printout of your graphs and label $f(x)$, $l(x)$ and $q(x)$)

Based on your graphs, which is a closer approximation to $y = f(x) = \sqrt[3]{x}$ for x near 1, $l(x)$ or $q(x)$?

To test this observation, use $l(x)$ and $q(x)$ to *approximate* $\sqrt[3]{1.3}$. Which is a closer approximation?

Now use $l(x)$ and $q(x)$ to *approximate* $\sqrt[3]{0.94}$. Which is a closer approximation?

Using the geometry of the tangent line and tangent parabola, why do you think one is necessarily a better approximation to a function than the other?

Exercise 2: **Best Linear and Quadratic Approximations**

As it turns out, the *best* linear approximation to a function $y = g(x)$ for x near a is the tangent line to the function at a, and the *best* quadratic approximation to a function $y = g(x)$ for x near a is the tangent parabola to the function at a:

$l(x) = g'(a)(x-a) + g(a)$ and $q(x) = \frac{g''(a)}{2}(x-a)^2 + g'(a)(x-a) + g(a)$

a) For this exercise, consider the function $g(x) = xe^{4-x}$. First find the tangent line and tangent parabola to $y = g(x)$ for x near 4 ($l(x)$ and $q(x)$).

$l(x) =$ $q(x) =$

b) Now consider a first degree polynomial $P_1(x)$ with an adjusted slope and a second degree polynomial $P_2(x)$ with an adjusted concavity as follows:

$$P_1(x) = (g'(a) + 0.2)(x-a) + g(a)$$

$$P_2(x) = \frac{(g''(a) - 0.2)}{2}(x-a)^2 + g'(a)(x-a) + g(a)$$

Find these two polynomials.

$P_1(x) =$ $P_2(x) =$

c) Graph the *function* and its *tangent line* and the *first degree polynomial* together in a graphing window centered at the point $(4, g(4))$, and answer the following questions: (include a printout of your graphs and label $g(x)$, $l(x)$ and $P_1(x)$)

d) Based on your graphs, which is a closer approximation to $y = g(x) = xe^{4-x}$ for x near 4, $l(x)$ or $P_1(x)$?

To test this observation, use $l(x)$ and $P_1(x)$ to approximate $g(4.2)$, $g(4.02)$, and $g(4.002)$. Which is better approximation?

e) Now graph the *function* and its *tangent parabola* and the *second degree polynomial* together in a graphing window centered at the point $(4, g(4))$, and answer the following questions: (include a printout of your graphs and label $g(x)$, $q(x)$ and $P_2(x)$)

f) Based on your graphs, which is a closer approximation to $y = g(x) = xe^{4-x}$ for x near 4, $q(x)$ or $P_2(x)$?

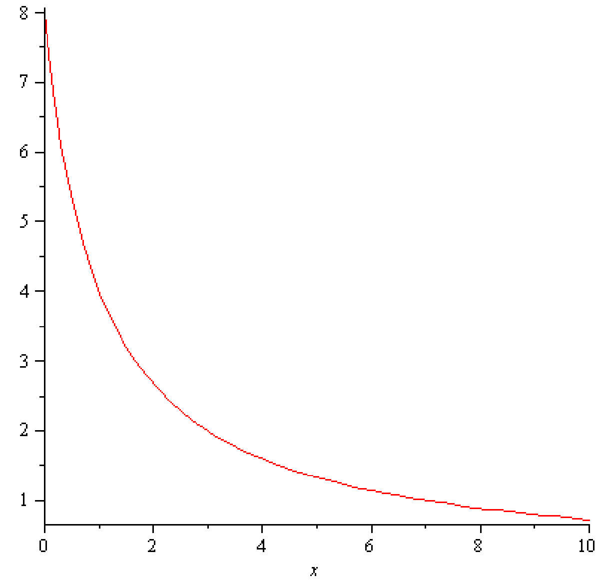
To test this observation, use $q(x)$ and $P_2(x)$ to approximate $g(4.2)$, $g(4.02)$, and $g(4.002)$. Which is better approximation?

Using the geometry of the derivative, why do you think the tangent line and the tangent parabola are the *best* linear and quadratic approximations to a function?

Exercise 3: **Overestimation and Underestimation**

For this exercise, the graphs of two functions are given. Use the geometry of the graphs to determine whether the tangent line to the graph at a point will be an overestimation or an underestimation to the function.

a) Given the following graph of $y = f(x)$, determine whether the tangent line at $x = 4$ is an overestimation or an underestimation to $y = f(x)$ for x near 4. Why?



b) Given the following graph of $y = g(x)$, determine whether the tangent line at $x = 4$ is an overestimation or an underestimation to $y = g(x)$ for x near 4. Why?