Newton's Method

Newton's Method is a technique for approximating real zeros of a function. It uses tangent lines to approximate the graph of the function near its x-intercepts.

Newton's Method assumes that f is continuous on $[a, b]$ and differentiable on the interval (a, b). If $f(a)$ and $f(b)$ differ in sign, then by the IVT, f must have at least one zero in the interval (a, b) which we can approximate by $x = x_1$.

Newton's Method is based on the assumption that the graph of f and the tangent line at $(x_1, f(x_1))$ both cross the x-axis at about the same point. Because you can easily calculate the x-intercept for this tangent line, you can use it as a second, better, estimate for the zero of f. This process gets repeated for as many iterations as needed to reach a certain degree of accuracy.

At $(x_1, f(x_1))$, the slope of the tangent line is $f'(x_1)$.
So the equation of the tangent line is:

To find the x-intercept of this tangent line, we want to let $y = 0$ and solve for x:

Call this new estimate x_2:

Newton's Method for Approximating the Zeros of a Function

Let $f(c) = 0$, where f is differentiable on an open interval containing c. Then, to approximate c, use the following steps.

1. Make an initial estimate x_1 that is close to c. (A graph is helpful.)

2. Determine a new approximation
$$x_{n+1} = x_n - \frac{f(x_{n})}{f'(x_n)}.$$

3. If $|x_n - x_{n+1}|$ is within the desired accuracy, let x_{n+1} serve as the final approximation. Otherwise, return to Step 2 and calculate a new approximation.

Each successive application of this procedue is called an **iteration**.

1. Calculate three iterations of Newton's Method for $f(x) = 2x^2 - 3$ using $x_1 = 1$.

2. Approximate the zero of the function $f(x) = x - 3 + lnx$.
   Continue the iterations until two successive approximations differ by less than 0.001.

Note: When the approximations approach a limit, the sequence $x_1, x_2, x_3, x_4, \ldots$ $, x_n, \ldots$ is said to converge. If the limit is c, it can be shown that c must be a zero of f.

Newton's Method does not always yied a convergent sequence:

Suppose that x_1 gives a horizontal tangent line, then $f'(x_1) = 0$ which would result in dividing by zero in Newton's Method.

If your x_1 does yield a horizontal tangent line, then usually you can just pick a different value for x_1.

3. Apply Newton's Method to $f(x) = 2sinx + cos2x$ with $x_1 = \frac{3\pi}{2}$.

4. Apply Newton's Method to $f(x) = x^{\frac{1}{3}}$ with $x_1 = 0.1$.

It can be shown that a condition sufficient to produce convergence of Newton's Method to a zero of f is that

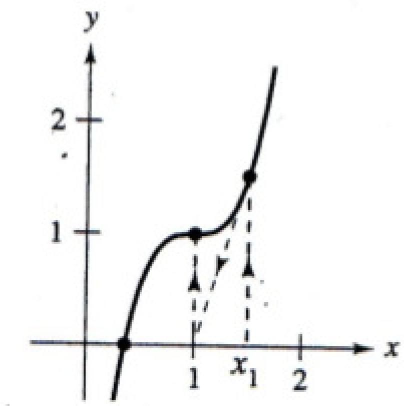
$$\left|\frac{f(x)f''(x)}{[f'(x)]^2}\right| < 1$$

on an open interval containing the zero.

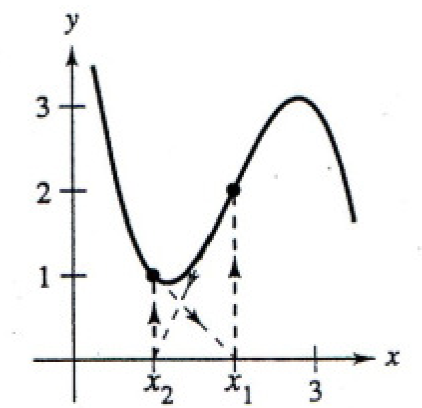
For example, in the last problem:

Apply Newton's Method using the given initial guess and explain why the method fails:

a) $y = 4x^3 - 12x^2 + 12x - 3,\ x_1 = \frac{3}{2}$



b) $f(x) = -x^3 + 6x^2 - 10x + 6,\ x_1 = 2$



c) $f(x) = 2\sin x + \cos 2x,\ x_1 = \frac{3\pi}{2}$