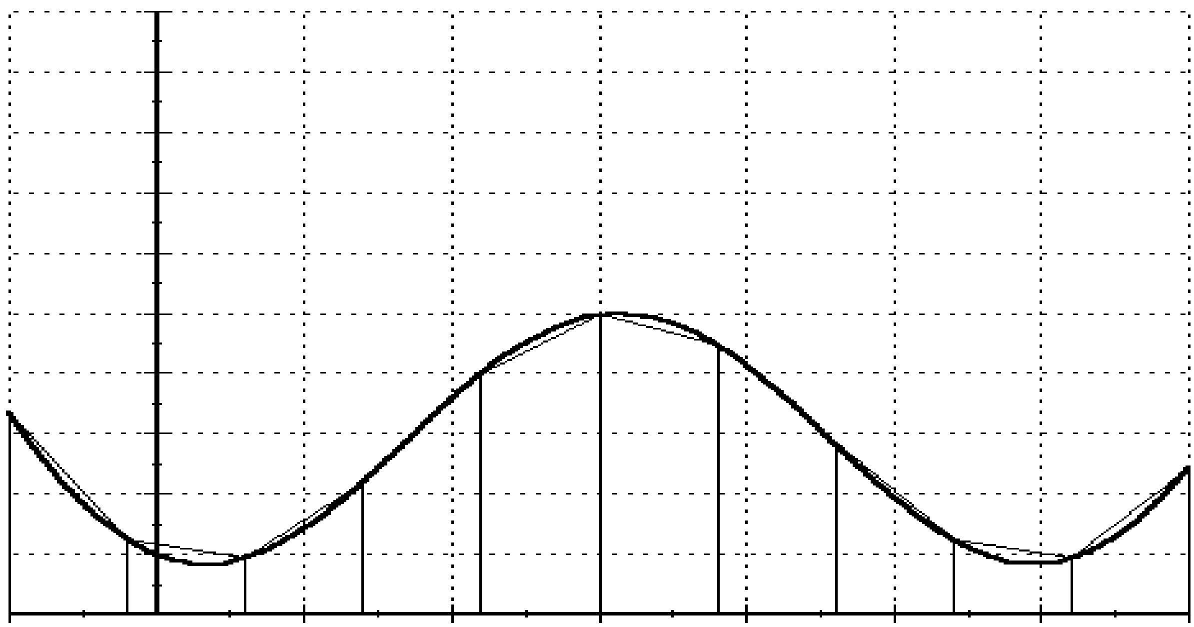
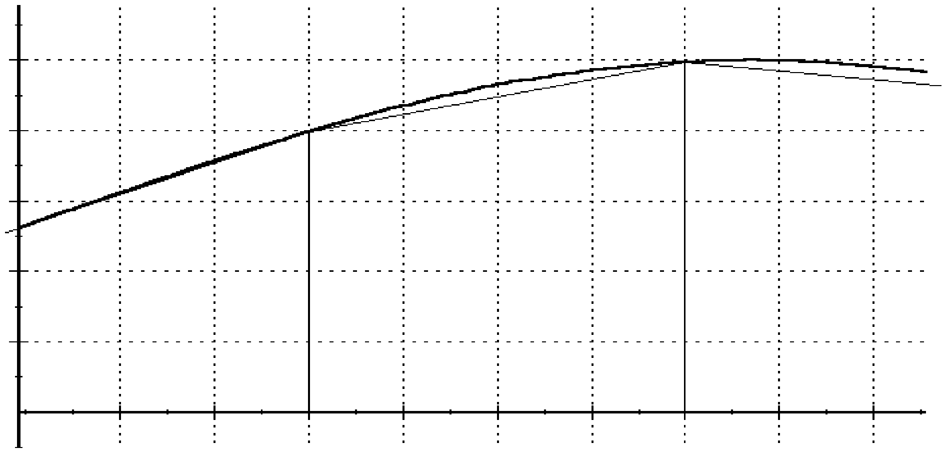
Numerical Integration

Many elementary functions do not have antiderivatives that are elementary functions. Functions such as $\sin x^2$, $\dfrac{\cos x}{x}$, and $\sqrt{1-x^2}$ are not the derivatives of any elementary functions. Since we do not know the antiderivative, we cannot apply the Fundamental Theorem of Calculus. Finding the sum of an infinite number of upper or lower sums most likely cannot be accomplished because we don't have summation formulas for sums such as $\sin i^2$ or $\sqrt{1-i^2}$. If we are going to use a finite number of areas, there are methods that are more accurate than summing rectangles.

We will look at two such methods. The first is called the Trapezoidal Rule.





Area of illustrated trapezoid is:

Area of ith trapezoid is:

Area of n trapezoids is:

If $f(x)$ is continuous, then the Trapezoidal Rule for approximating an integral is given by

$$\int_a^b f(x)\,dx \approx \frac{b-a}{2n}(f(x_0) + 2f(x_1) + 2f(x_2) + \ldots + 2f(x_{n-1}) + f(x_n))$$

Note: If we take the limit as $n \to \infty$ the right side approaches $\int_a^b f(x)\,dx$

Summation notation for Trapezoidal Rule:

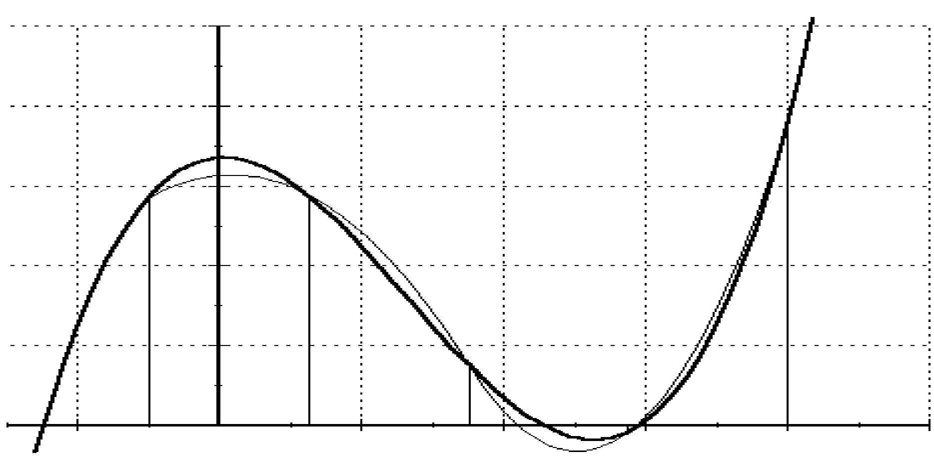
Summation notation for Midpoint Rule:

In approximating integrals using upper and lower sums we are approximating the function $f(x)$ by a constant function in each subinterval.

Using the Trapezoidal Rule we are approximating $f(x)$ by a first degree function.

A second approximation method uses a second degree function to approximate the value of the function $f(x)$. This method is named after an English mathematician Thomas Simpson.

Recall that there is a parabola that passes through any 3 points that have different x-coordinates if they are not on a straight line.



Simpson's Rule: If $f(x)$ is continuous, then the Simpson's Rule for approximating an integral is given by (n must be even)

$$\int_a^b f(x)\,dx \approx \frac{b-a}{3n}\left(f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+\ldots+4f(x_{n-1})+f(x_n)\right)$$

Note: If we take the limit as $n \to \infty$ the right side approaches $\int_a^b f(x)\,dx$

<u>Errors:</u> $E_T \leq \frac{(b-a)^3}{12n^2}[\max|f''(x)|],\ a \leq x \leq b$

$$E_S \leq \frac{(b-a)^5}{180n^4}\left[\max\left|f^{(4)}(x)\right|\right],\ a \leq x \leq b$$

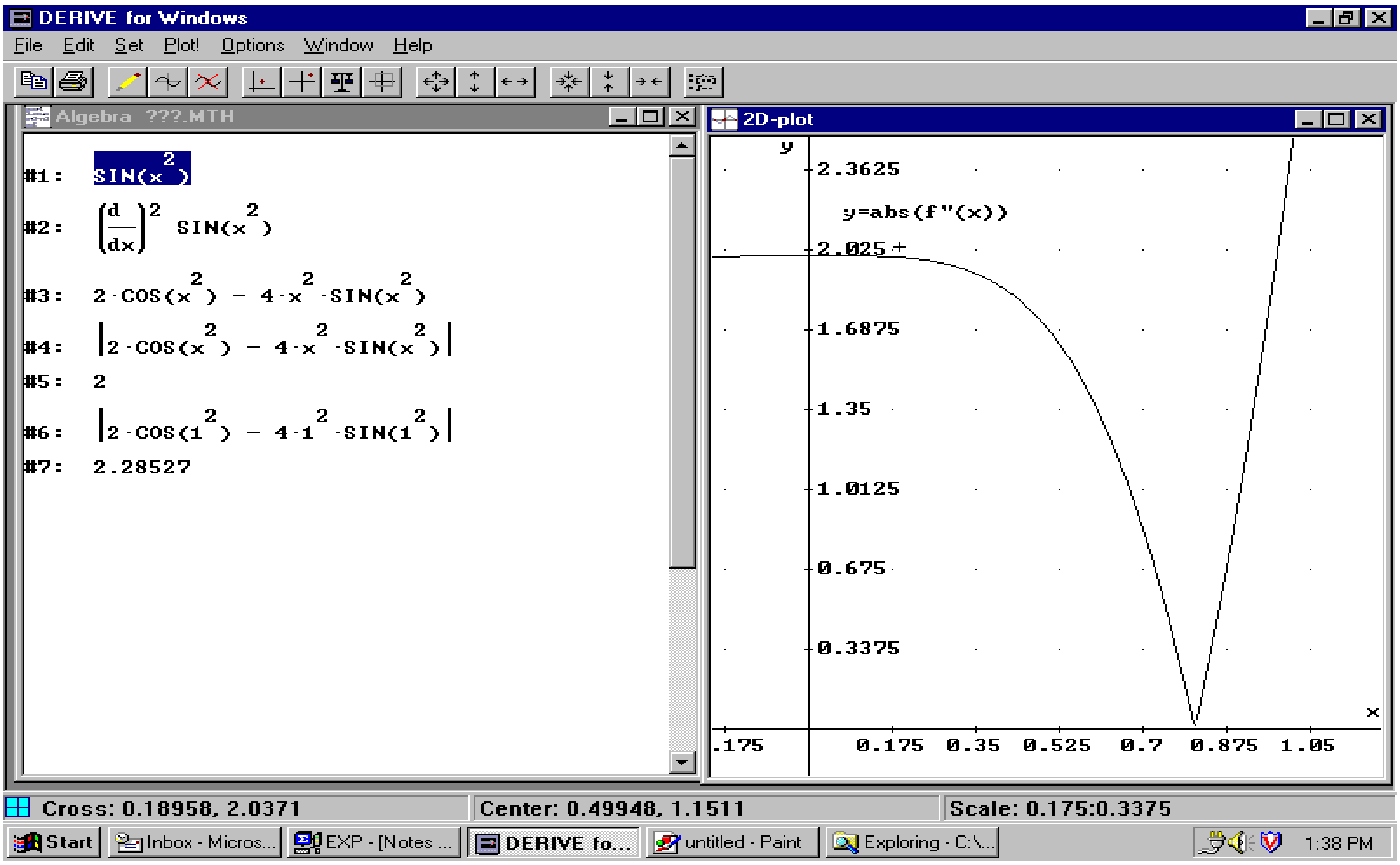
Examples:

1. Use Trapezoidal Rule and Simpson's Rule with $n = 4$ to approximate
$$\int_0^{\frac{\pi}{2}} \sqrt{1+\cos^2 x}\, dx$$

$$\int_0^{\frac{\pi}{2}} \sqrt{1+\cos^2 x}\, dx \approx T = \frac{b-a}{2n}\left(f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)\right)$$

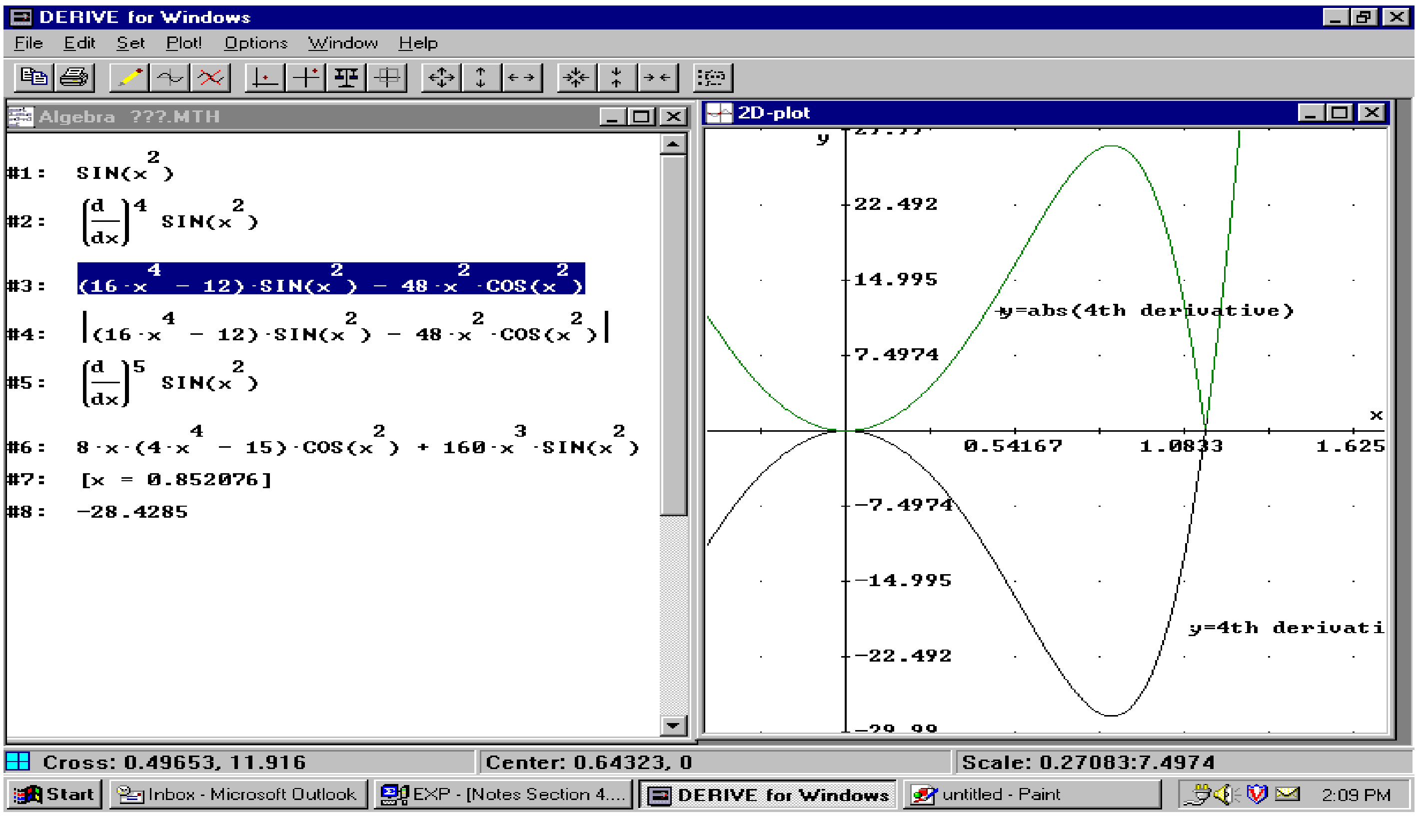
$$\int_0^{\frac{\pi}{2}} \sqrt{1+\cos^2 x}\, dx \approx S = \frac{b-a}{3n}\left(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)\right)$$

2. Find the number of partition points so that the error using Trapezoidal Rule to approximate $\int_0^1 \sin x^2\, dx$ is $< .00001$



$$E_T \le \frac{(b-a)^3}{12n^2}\left[\max|f''(x)|\right],\ a \le x \le b$$

3. Find the number of partition points so that the error using Simpson's Rule to approximate $\int_0^1 \sin x^2\,dx$ is $< .00001$



$$E_S \le \frac{(b-a)^5}{180n^4}\left[\max\left|f^{(4)}(x)\right|\right],\ a \le x \le b$$