One-sided Limits and Continuity

One-Sided Limits:

Right-Hand Limit: $\lim_{x \to a^+} f(x) = L$ means that $f(x) \to L$ as $x \to a$ from the right ($x > a$)

Left-Hand Limit: $\lim_{x \to a^-} f(x) = L$ means that $f(x) \to L$ as $x \to a$ from the left ($x < a$)

Theorem: The ordinary limit $\lim_{x \to a} f(x)$ exists if and only if both corresponding one-sided limits exist and are equal.

$$\lim_{x \to a} f(x) = L \Leftrightarrow \lim_{x \to a^+} f(x) = L \Leftrightarrow \lim_{x \to a^-} f(x) = L$$

Example: $f(x) = \sqrt{x-4}$

$\lim_{x \to 4^+} \sqrt{x-4} =$ $\lim_{x \to 4^-} \sqrt{x-4} =$ $\lim_{x \to 4} \sqrt{x-4} =$

All limit theorems hold.

$$\lim_{x \to 2^+} \frac{2-x}{x^2-2}$$

$$\lim_{x \to 4^-} \frac{\sqrt{x}-2}{x-4}$$

$$\lim_{x \to 4^-} \frac{|x-4|}{x-4}$$

$$\lim_{x \to 1^+} \frac{|x-1|}{1-x}$$

$$\lim_{x \to 0^-} [[x]]$$

$$\lim_{x \to 1^+} [[x+2]]$$

$$\lim_{x \to 2^+} (2x - [[x]])$$

$$\lim_{x \to 1} f(x) = \begin{cases} 3x - 1 & , \quad x < 1 \\ x^2 & , \quad x > 1 \end{cases}$$

$$\lim_{x \to 1^-} f(x) =$$

$$\lim_{x \to 1^+} f(x) =$$

$$\lim_{x \to 1} f(x) =$$

$$\lim_{x \to 2} \text{ if } f(x) = \begin{cases} x^2 & , \quad x < 2 \\ 3x - 2 & , \quad x \geq 2 \end{cases}$$

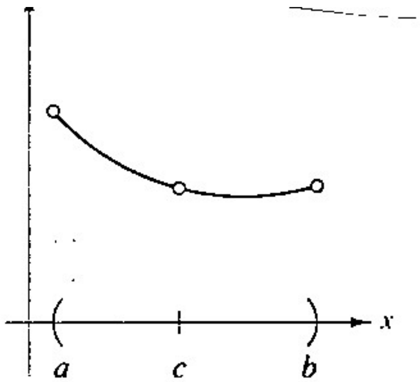
$$\lim_{x \to 2^-} f(x) =$$

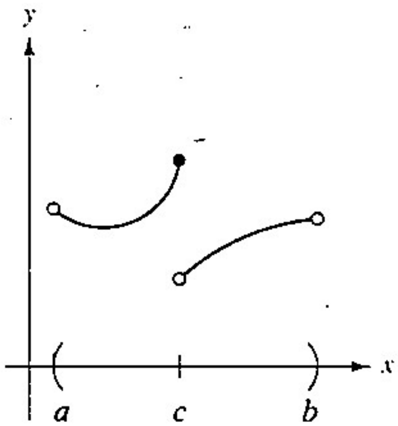
$$\lim_{x \to 2^+} f(x) =$$

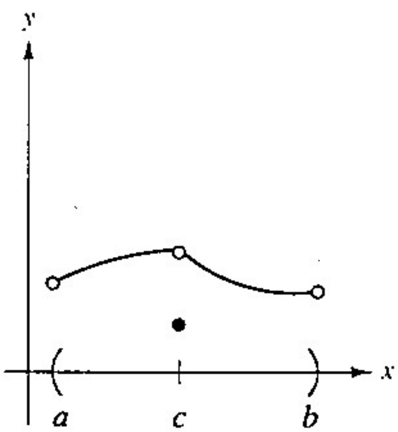
$$\lim_{x \to 2} f(x) =$$

Continuity:

A function is continuous at $x = c$ if it has no holes, jumps or gaps at c.







Continuity at a Point: A function f is continuous at a point c if the following three conditions exist:

1.

2.

3.

Continuity on an Interval: A function f is continuous on an open interval (a, b) if it is continuous at each point in the interval. A function that is continuous on the entire real line $(-\infty,\ +\infty)$ is said to be "everywhere" continuous.

Consider a function such as:

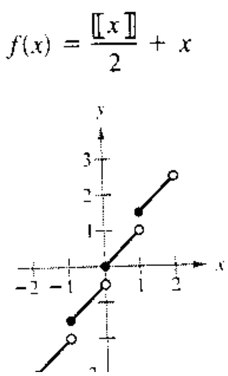
What would you say about the continuity of this function?

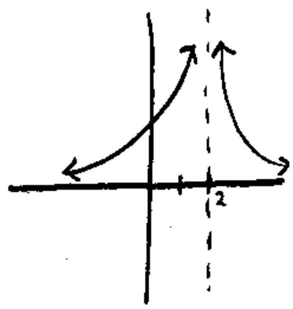
A function f is continuous on the closed interval $[a, b]$ if it is continuous at each point on the open interval (a, b) and if $\lim_{x \to a^+} f(x) = f(a)$ (f is continuous from the right at a) and $\lim_{x \to b^-} f(x) = f(b)$ (f is continuous from the left at b).

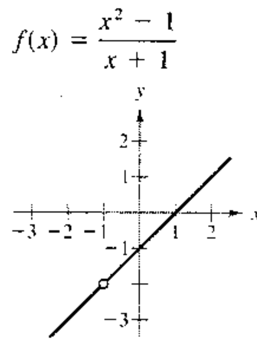
<u>There are two classifications of a point of discontinuity on a function.</u>

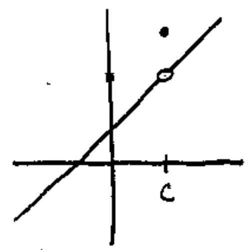
1. <u>Removable</u>: if $\lim_{x \to a} f(x)$ exists; function can be redefined at a to remove the discontinuity.

2. <u>Non-removable:</u> if $\lim_{x \to a} f(x)$ does not exist.









<u>Properties of Continuity</u>

If b is a real number and f and g are continuous at $x = c$, then the following functions are also continuous at c.

1. A scalar multiple: bf

2. A sum or difference: $f \pm g$

3. A product: fg

4. A quotient: $\frac{f}{g}$

The following types of functions are continuous at every point in their domains.

1. Polynomial 2. Rational 3. Radical 4. Trigonometric

Find $x-$ values (if any) at which f is not continuous. Classify each discontinuity as removable or non-removable. If removable, redefine the function to remove the discontinuity.

$f(x) = \dfrac{1}{x^2+1}$

$f(x) = \dfrac{2}{x-1}$

$$f(x) = \frac{x-3}{x^2-9}$$

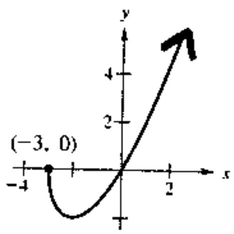
$$f(x) = \frac{|x-3|}{x-3}$$

$$f(x) = \begin{cases} -2x & \text{if} \quad x \leq 2 \\ x^2 - 4x + 1 & \text{if} \quad x > 2 \end{cases}$$

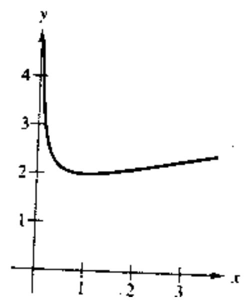
$$f(x) = \begin{cases} \tan \frac{\pi x}{4} & \text{if} \quad |x| < 1 \\ x & \text{if} \quad |x| \geq 1 \end{cases}$$

State the intervals of continuity:

58. $f(x) = x\sqrt{x+3}$



60. $f(x) = \dfrac{x+1}{\sqrt{x}}$

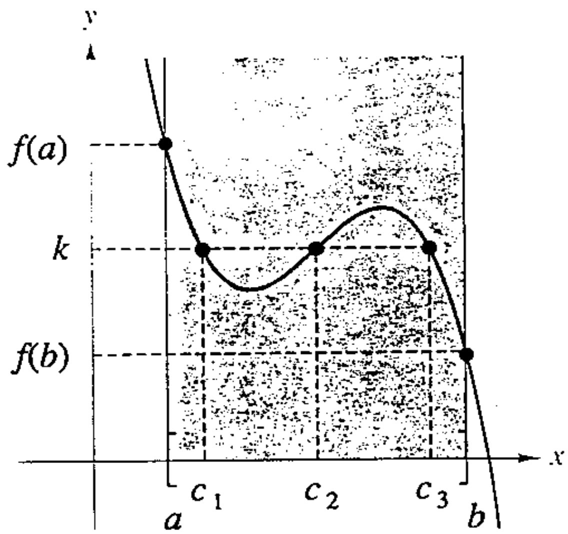


Find "a" so that f is continuous for all real numbers x.

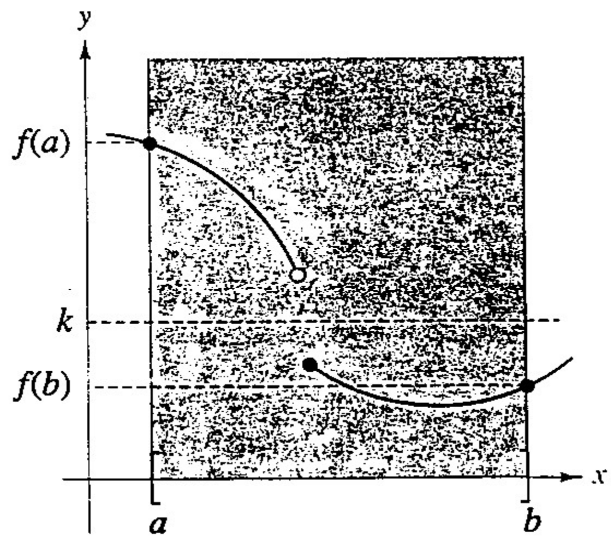
$$f(x) = \begin{cases} \dfrac{x^2 - a^2}{x - a} & \text{if} \quad x \neq a \\ 8 & \text{if} \quad x = a \end{cases}$$

Intermediate Value Theorem

If f is continuous on the closed interval $[a, b]$ and k is any number between $f(a)$ and $f(b)$, then there is at least one number c in $[a, b]$ such that $f(c) = k$.



f is continuous on $[a, b]$.
(For k, there exist 3 c's.)



f is not continuous on $[a, b]$.
(For k, there are no c's.)

A continuous function on $[a, b]$ takes on all values between $f(a)$ and $f(b)$.

Use the Intermediate Value Theorem to show that $f(x) = x^3 + 2x - 1$ has a zero in the interval $[0, 1]$.

Use the Intermediate Value Theorem to prove that $\sqrt{3}$ is between 1 and 2.