Product and Quotient Rules and Higher-Order Derivatives

1. Product Rule: If $f(x)$ and $g(x)$ are differentiable functions, then $\frac{d}{dx}(f(x) \cdot g(x)) =$

The derivative of a sum is the sum of the derivatives. Is the derivative of a product equal to the product of the derivatives?

Consider $f(x) = x^2 \cdot x^5$

Proof: $$\frac{d}{dx}(f \cdot g)(x) = \lim_{\Delta x \to 0} \frac{(f \cdot g)(x + \Delta x) - (f \cdot g)(x)}{\Delta x}$$

$$\boldsymbol{\frac{d}{dx}(f \cdot g)(x) = f(x)g'(x) + g(x)f'(x)}$$

A formula for $\frac{d}{dx}(f(x) \cdot g(x) \cdot h(x)) =$

2. Quotient Rule: If $f(x)$ and $g(x)$ are differentiable functions, then

$$\frac{\boldsymbol{d}}{\boldsymbol{dx}}\left(\frac{\boldsymbol{f(x)}}{\boldsymbol{g(x)}}\right) = \frac{\boldsymbol{g(x)f'(x) - f(x)g'(x)}}{\boldsymbol{(g(x))^2}}$$

Examples: Find each derivative

1. $f(x) = (x^2 - 2x + 1)(x^3 - 1)$

2. $f(x) = \dfrac{x^3 + 3x + 2}{x^2 - 1}$

3. $g(x) = \sqrt[3]{x}\left(\sqrt{x} + 3\right)$

4. $f(x) = \dfrac{x(x^2 - 1)}{x + 3}$

Avoid using the quotient rule whenever possible:

5. $f(x) = \dfrac{4x^{\frac{3}{2}}}{x}$

6. $f(x) = \dfrac{4}{5x^2}$

7. $f(x) = \dfrac{2x^3 - x}{7}$

8. $f(x) = \dfrac{3}{x^8}$

Power Rule for Negative Exponents:

$\dfrac{d}{dx}(x^n) = nx^{n-1}$ when n is a negative integer.

Theorem: $\mathsf{D}_x(\tan x) =$

Theorem: $\mathsf{D}_x(\sec x) =$

Theorem: $\mathsf{D}_x(\cot x) = -\csc^2 x$

Theorem: $\mathsf{D}_x(\csc x) = -\csc x \cot x$

Find each derivative:

9. $f(x) = \dfrac{\sec x}{x}$

10. $f(x) = x\csc x + \cot x$

11. $f(x) = \dfrac{1 - \cos x}{\sin x}$

12. $f(x) = \csc x - \cot x$

13. Find the equation of the tangent line to the graph of $f(x) = \tan x$ at $\left(\dfrac{\pi}{4}, 1\right)$.

14. Find where the horizontal tangent line(s) occur for $f(x) = \frac{x^2}{x^2+1}$.

Higher order derivatives:

Notation:

First derivative

Second derivative

Third derivative

Fourth derivative

nth derivative

Applications of higher order derivatives:

position function: $s(t)$
velocity function: $v(t) = s'(t)$
acceleration function: $a(t) = v'(t) = s''(t)$

15. Find $f''(x)$ if $f(x) = \sec x$

16. If $y' = 2 - \dfrac{2}{x}$, find $\dfrac{d^4y}{dx^4}$

17. An astronaut standing on the moon throws a rock into the air. The height of the rock is given by $s(t) = -\frac{27}{10}t^2 + 27t + 6$ where s is measured in feet and time is measured in seconds.

a) Find a velocity function.

b) Find an acceleration function.

c) Find the time when the rock is at its highest point.

d) How does the acceleration of the rock compare with acceleration on earth?

18. Let $s(t) = -16t^2 + 32t + 48$ represent the height of an object propelled from a cliff 48 feet high. Find the velocity and acceleration when $t = 2$. Also find when the object reaches its maximum height. What is the maximum height?