

Product and Quotient Rules and Higher-Order Derivatives

1. Product Rule: If $f(x)$ and $g(x)$ are differentiable functions, then
- $$\frac{d}{dx}(f(x) \cdot g(x)) =$$

The derivative of a sum is the sum of the derivatives. Is the derivative of a product equal to the product of the derivatives?

Consider $f(x) = x^2 \cdot x^5$

Proof:
$$\frac{d}{dx}(f \cdot g)(x) = \lim_{\Delta x \rightarrow 0} \frac{(f \cdot g)(x + \Delta x) - (f \cdot g)(x)}{\Delta x}$$

$$\frac{d}{dx}(f \cdot g)(x) = f(x)g'(x) + g(x)f'(x)$$

A formula for $\frac{d}{dx}(f(x) \cdot g(x) \cdot h(x)) =$

2. Quotient Rule: If $f(x)$ and $g(x)$ are differentiable functions, then

$$\frac{d}{dx} \left(\frac{f(x)}{g(x)} \right) = \frac{g(x)f'(x) - f(x)g'(x)}{(g(x))^2}$$

Examples: Find each derivative

1. $f(x) = (x^2 - 2x + 1)(x^3 - 1)$

$$2. \quad f(x) = \frac{x^3 + 3x + 2}{x^2 - 1}$$

$$3. \quad g(x) = \sqrt[3]{x}(\sqrt{x} + 3)$$

$$4. \quad f(x) = \frac{x(x^2 - 1)}{x + 3}$$

Avoid using the quotient rule whenever possible:

5. $f(x) = \frac{4x^{\frac{3}{2}}}{x}$

6. $f(x) = \frac{4}{5x^2}$

7. $f(x) = \frac{2x^3 - x}{7}$

8. $f(x) = \frac{3}{x^8}$

Power Rule for Negative Exponents:

$$\frac{d}{dx}(x^n) = nx^{n-1} \text{ when } n \text{ is a negative integer.}$$

Theorem: $D_x(\tan x) =$

Theorem: $D_x(\sec x) =$

Theorem: $D_x(\cot x) = -\csc^2 x$

Theorem: $D_x(\csc x) = -\csc x \cot x$

Find each derivative:

9. $f(x) = \frac{\sec x}{x}$

10. $f(x) = x \csc x + \cot x$

11. $f(x) = \frac{1 - \cos x}{\sin x}$

12. $f(x) = \csc x - \cot x$

13. Find the equation of the tangent line to the graph of $f(x) = \tan x$ at $\left(\frac{\pi}{4}, 1\right)$.

14. Find where the horizontal tangent line(s) occur for $f(x) = \frac{x^2}{x^2+1}$.

Higher order derivatives:

Notation:

First derivative

Second derivative

Third derivative

Fourth derivative

*n*th derivative

Applications of higher order derivatives:

position function: $s(t)$

velocity function: $v(t) = s'(t)$

acceleration function: $a(t) = v'(t) = s''(t)$

15. Find $f''(x)$ if $f(x) = \sec x$

16. If $y' = 2 - \frac{2}{x}$, find $\frac{d^4y}{dx^4}$

17. An astronaut standing on the moon throws a rock into the air. The height of the rock is given by $s(t) = -\frac{27}{10}t^2 + 27t + 6$ where s is measured in feet and time is measured in seconds.
- a) Find a velocity function.
 - b) Find an acceleration function.
 - c) Find the time when the rock is at its highest point.
 - d) How does the acceleration of the rock compare with acceleration on earth?

- 18.** Let $s(t) = -16t^2 + 32t + 48$ represent the height of an object propelled from a cliff 48 feet high. Find the velocity and acceleration when $t = 2$. Also find when the object reaches its maximum height. What is the maximum height?