Riemann Sums and Definite Integrals

1. Find the area of the region bounded by $y = \sqrt{x}$ on the interval $[0, 1]$.
Use a regular partition:

Use a partition where the right endpoint of the ith subinterval is given by $x_i = \dfrac{i^2}{n^2}$ and Δx_i is the width of the ith subinterval.

We see, at least in this example, that it is not necessary to have subintervals of equal length.

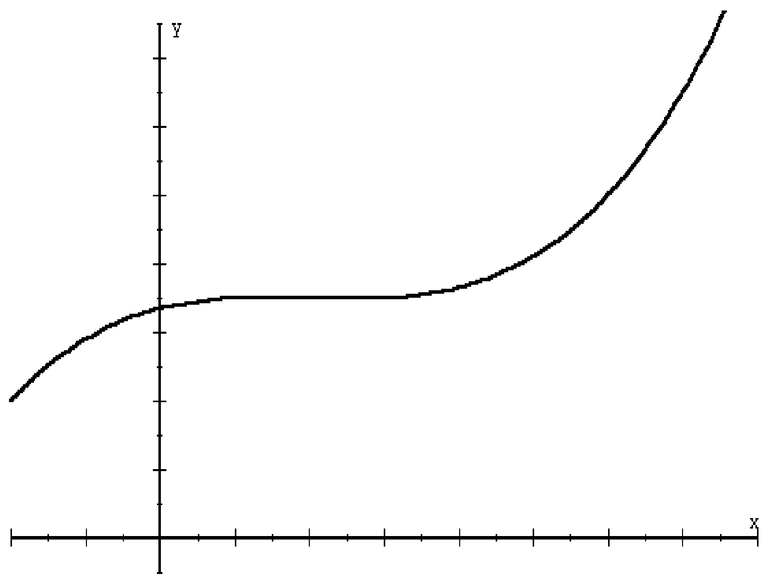
Definition of a Riemann Sum: Let $f(x)$ be defined on a closed interval $[a, b]$, and let Δ be a partition of $[a, b]$ given by $a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b$, where Δx_i is the length of the ith subinterval. If c_i is ANY point in the ith subinterval, then the sum of $\sum_{i=1}^{n} f(c_i)\Delta x_i, \;\; x_{i-1} \leq c_i \leq x_i,$ is called a **Riemann Sum** of $f(x)$ for the partition Δ.

2. Find the Riemann sum for $f(x) = \sin x$ over the interval $[0, 2\pi]$ where $x_0 = 0$, $x_1 = \dfrac{\pi}{4}$, $x_2 = \dfrac{\pi}{3}$, $x_3 = \pi$, $x_4 = 2\pi$ and $c_1 = \dfrac{\pi}{6}, \;\; c_2 = \dfrac{\pi}{3}, \;\; c_3 = \dfrac{2\pi}{3}, \;\; c_4 = \dfrac{3\pi}{2}$.

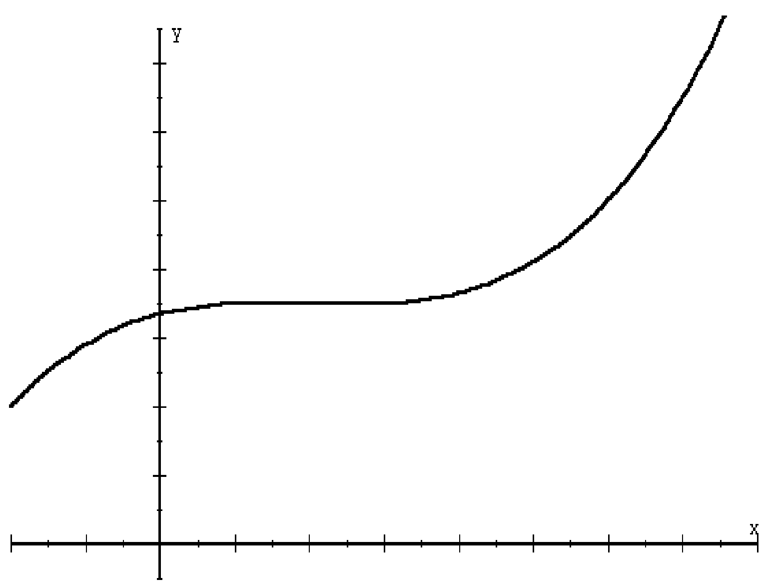
Under what conditions will a Riemann Sum approximate the area?

If $f(x)$ is continuous and non-negative on $[a, b]$, a Riemann sum can be used to estimate area under the graph of $f(x)$.

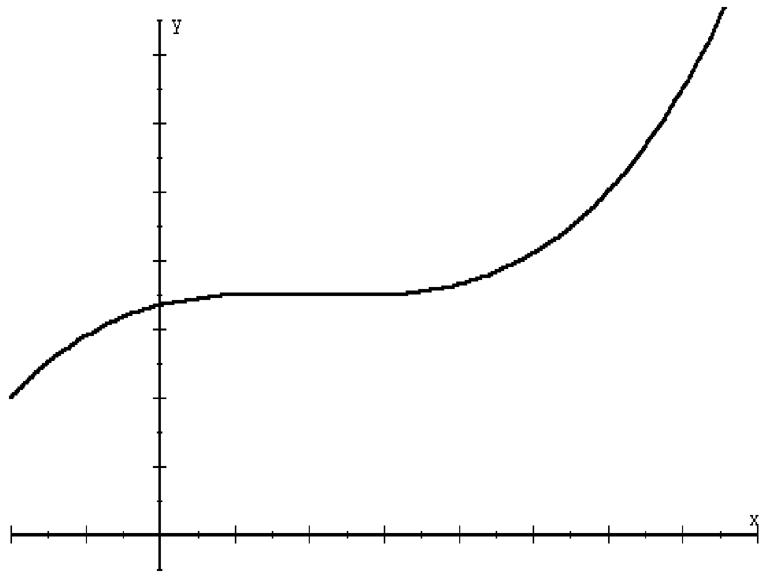
If we choose a regular (equal length subintervals) partition, how can we find the actual area?



What if the partition is irregular and $n \to \infty$?



In order for a Riemann sum to approximate area we need to require that



The length of the largest subinterval of a partition Δ is called the norm of the partition and is denoted by $\|\Delta\|$. If every subinterval is of equal length, the partition is regular and the norm is $\|\Delta\| = \Delta x = \dfrac{b-a}{n}$. Thus if we have a regular partition the norm of the partition approaching 0 is equivalent to n approaching infinity. If the partition is not regular, the equivalence does not hold.

Definition of a Definite Integral: If f is defined on the closed interval $[a, b]$ and the limit

$$\lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i$$

exists, then f is integrable on $[a, b]$ and the limit is

$$\lim_{\|\Delta\|\to 0} \sum_{i=1}^{n} f(c_i)\Delta x_i = \int_a^b f(x)dx.$$

The limit is called the definite integral of f from a to b.
a is the lower limit of integration.
b is the upper limit of integration.

We have two notations that look somewhat similar but at this point are totally unrelated.

Definite integral: $\displaystyle\int_a^b x^2 dx$ represents

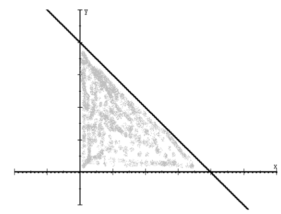
Indefinite integral: $\displaystyle\int x^2 dx$ represents

Theorem: If a function $f(x)$ is continuous on a closed interval $[a, b]$, then $f(x)$ is integrable on $[a, b]$.

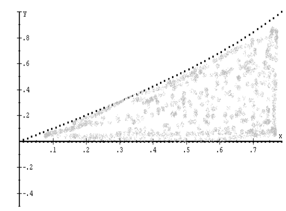
3. Evaluate $\int_{-1}^{2} x^3 dx$

4. Set up an integral that yields the area of the shown region.

a) $f(x) = 4 - 2x$



b) $g(x) = \tan x$



5. Sketch the region whose area is indicated by the definite integral and use a geometric formula to evaluate the integral.

a) $\displaystyle\int_0^4 \frac{x}{2}\,dx$

b) $\displaystyle\int_{-a}^{a}(a-|x|)dx\; a>0$

c) $\displaystyle\int_{-2}^{2}\sqrt{4-x^2}dx$

Properties of Definite Integrals:

1. $\int_a^a f(x)dx = 0$

2. $\int_a^b f(x)dx = -\int_b^a f(x)dx$

3. $\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$

4. $\int_a^b kf(x)dx = k\int_a^b f(x)dx$

5. $\int_a^b [f(x) \pm g(x)]dx = \int_a^b f(x)dx \pm \int_a^b g(x)dx$

6. If $f(x) \leq g(x)$ on $[a,b]$, then $\int_a^b f(x)dx \leq \int_a^b g(x)dx$.

6. Given $\int_{-1}^{1} f(x)dx = 0$ and $\int_{0}^{1} f(x)dx = 5$, find:

a) $\int_{-1}^{0} f(x)dx$

b) $\int_{0}^{1} f(x)dx - \int_{-1}^{0} f(x)dx$

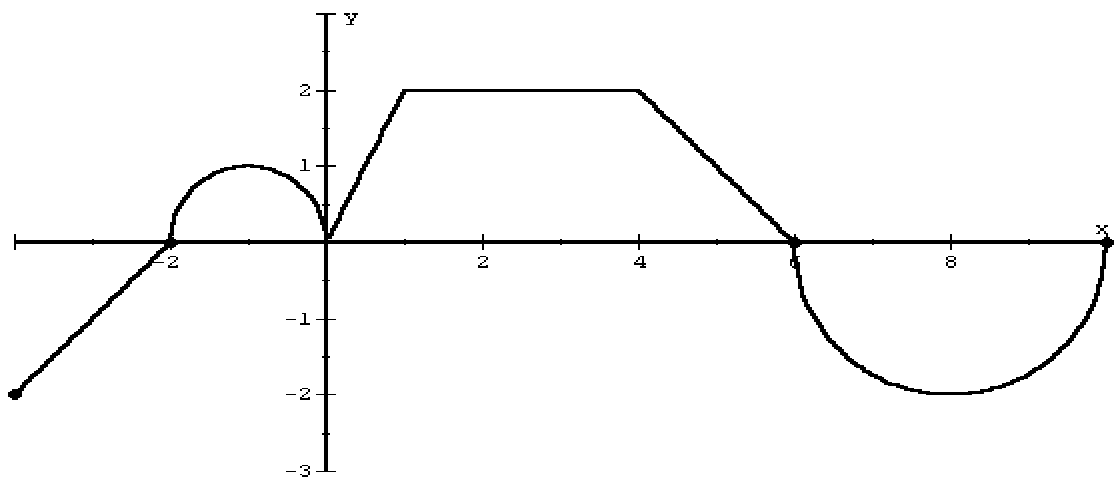
c) $\int_{-1}^{1} (2f(x) + 3)dx$

d) $\int_{1}^{0} 2f(x)dx$

7. Express the limit as a definite integral.

$$\lim_{\|\Delta\|\to 0}\sum_{i=1}^{n}\frac{3}{c_i^2}\Delta x_i \text{ on } [1,3]$$

8. Assume the region below is bounded by semicircles and straight lines. Find the value of the indicated definite integrals. Which integrals can be interpreted as area?



a) $\int_{-2}^{1} f(x)\,dx$

b) $\int_{10}^{3} f(x)\,dx$

c) $\int_{-4}^{0} f(x)\,dx$

d) $\int_{3}^{8} f(x)\,dx$