Rolle's Theorem and Mean Value Theorem

Rolle's Theorem: Let f be continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b). If $f(a) = f(b)$, then there is at least one number c in (a, b) such that $f'(c) = 0$.

The hypotheses of Rolle's Theorem that guarantee at least one value of c where $f'(c) = 0$ are:

1.

2.

3.

The following examples show why each of these conditions are necessary.

Example **1**. f fails to be continuous on $[a, b]$

Example **2**. f fails to be differentiable on (a, b)

Example **3**. $f(a) \neq f(b)$

If one of the three hypotheses for Rolle's Theorem fails to be true, there might exist a number c for which $f'(c) = 0$, but there is no guarantee.

In each of the following determine whether Rolle's Theorem can be applied to f on the indicated interval. If it can't be applied, state why. If it can be applied, find all values of c in the interval for which Rolle's Theorem applies.

4. $f(x) = 3 - |x - 3|$ on $[0, 6]$

5. $g(x) = \dfrac{x^2 - 1}{x}$ on $[-1, 1]$

6. $f(x) = 4x - \tan \pi x$ on $\left[-\dfrac{1}{4}, \dfrac{1}{4}\right]$

Mean Value Theorem: This theorem is a generalization of Rolle's Theorem. The hypotheses are the same except that we remove the requirement that $f(a) = f(b)$.

Mean Value Theorem: If f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then there exists a number c in (a, b) such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$

Paraphase: Geometrically, the MVT guarantees the existence of a tangent line that is parallel to the secant line through the points $(a, f(a))$ and $(b, f(b))$.

Another paraphrase: In terms of rates of change, the MVT implies that there must be a point in the open interval (a, b) at which the instantaneous rate of change is equal to the average rate of change over the interval $[a, b]$.

Note: This is exactly what Rolle's Theorem says, but with the added assumption that $f(a) = f(b)$ in which case the line joining the endpoints always has zero slope.

In each of the following determine whether the Mean Value Theorem can be applied to f on the indicated interval. If it can't be applied, state why. If it can be applied, find all values of c in the interval for which Mean Value Theorem applies.

7. $f(x) = |x - 2|$ on $[0, 4]$

8. $g(x) = \dfrac{1}{x - 1}$ on $[0, 2]$

9. $f(x) = x - 2\sin x$ on $[-\pi, \pi]$

Mean Value Theorem: If f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then there exists a number c in (a, b) such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$

Proof: For any x in the interval $[a, b]$, let $g(x)$ be the vertical distance from the secant line joining points P and Q to the graph of $f(x)$.

equation of line joining P & Q :

$g(x) =$

10. Use Rolle's Theorem to prove that the equation $4x^5 + 3x^3 + 3x - 2 = 0$ has exactly one root in the interval $(0, 1)$.

11. If $f'(x) = 0\ \forall x \in (a, b)$, then $f(x)$ is constant on $[a, b]$.

15. Can you find a function f such that $f(-2) = -2$, $f(2) = 6$ and $f'(x) < 1 \, \forall x$? Why or why not?

16. Find bounds on $f(3)$ if $f(0) = 0$ and $-2 \leq f'(x) \leq 2 \, \forall x$.