Summation Notation and Area

The sum of n terms $a_1, a_2,\ a_3,\ \ldots, a_n$ is written $\sum_{i=1}^{n} a_i = a_1 + a_2 +\ a_3 +\ \ldots + a_n.$ i is the index of summation, a_i is the ith term of the sum and the upper and lower bounds are 1 and n.

Examples:

1. $\sum_{i=3}^{6} i^2 =$

2. $\sum_{k=3}^{6} k^2 =$

3. $\sum_{k=1}^{n} n(k+1) =$

4. $\sum_{i=1}^{n} f(x_i)\Delta x =$

5. Write in summation notation:

$$\left[1 - \left(\frac{2}{n} - 1\right)^2\right]\left(\frac{2}{n}\right) + \left[1 - \left(\frac{2 \cdot 2}{n} - 1\right)^2\right]\left(\frac{2}{n}\right) + \ldots + \left[1 - \left(\frac{2 \cdot n}{n} - 1\right)^2\right]\left(\frac{2}{n}\right)$$

6. Write in summation notation:

$$\left[1-\left(\frac{1}{4}\right)^2\right]+\left[1-\left(\frac{2}{4}\right)^2\right]+\ldots+\left[1-\left(\frac{4}{4}\right)^2\right]$$

Summation Formulas

1. $\sum_{i=1}^{n} c =$

2. $\sum_{i=1}^{n} i =$

3. $\sum_{i=1}^{n} i^2$

4. $\sum_{i=1}^{n} i^3$

Properties of Summation:

1. $\sum_{i=1}^{n} k a_i =$

2. $\sum_{i=1}^{n} (a_i \pm b_i) =$

Examples:

7. $\sum_{i=1}^{15} (2i - 3)$

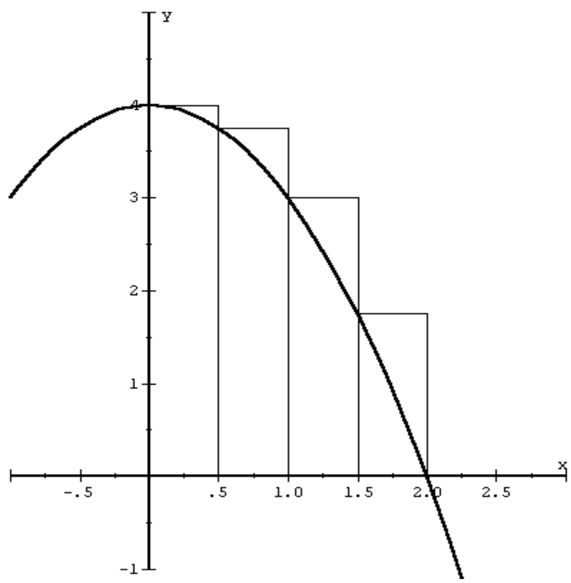
8. $\sum_{i=1}^{10} (i^2 - 1)$

9. Find the sum in terms of n, then find $\lim_{n\to\infty}$ sum

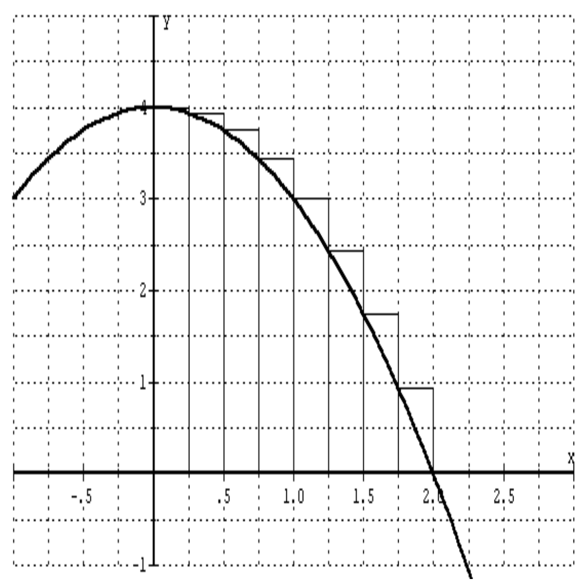
$$\sum_{i=1}^{n} \left(1 + \frac{2i}{n}\right)^{2} \left(\frac{2}{n}\right)$$

10. Approximate the area of the region bounded by the curve $f(x) = 4 - x^2$, the x-axis and the y-axis.

Use 4 rectangles:

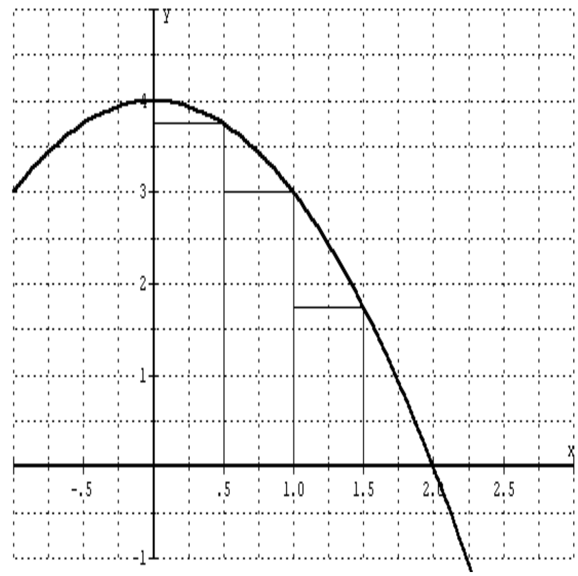


How could we improve upon our approximation?



Would this be an underestimate or an overestimate of the actual area?

Approximate the area again.



Would this be an underestimate or an overestimate?

To generalize:

To approximate the area bounded by $f(x)$, the x=axis and the vertical lines $x = a$ and $x = b$, begin by partitioning the interval into n subintervals:

Width = $\Delta x = \frac{b-a}{n}$

Partition $=$
$[a, a+\Delta x], [a+\Delta x, a+2\Delta x], [a+2\Delta x, a+3\Delta x], \ldots [a+(n-1)\Delta x, b]$

Let $f(m_i) =$ Minimum value of $f(x)$ in the ith subinterval
Let $f(M_i) =$ Maximum value of $f(x)$ in the ith subinterval

The inscribed rectangle lies inside the ith region

The circumscribed rectangle lies outside the ith region

(Area of inscribed rectangle)	$\leq$	(Area of circumscribed rectangle)
$\boldsymbol{f(m_i)\Delta x}$	$\leq$	$\boldsymbol{f(M_i)\Delta x}$

Lower Sum $\boldsymbol{= s(n) = \sum_{i=1}^{n} f(m_i)\Delta x}$ **area of inscribed rectangles**

Upper Sum $\boldsymbol{= S(n) = \sum_{i=1}^{n} f(M_i)\Delta x}$ **area of circumscribed rectangles**

The actual area of the region lies between these two sums:

$s(n) \leq$ Area of region $\leq S(n)$

11. Find $s(n)$ and $S(n)$ on the interval $[0, 2]$ for the function $f(x) = x^2$

Theorem: Let f be continuous and nonnegative on the interval $[a, b]$. The limits as $n \to \infty$ of both the lower and upper sums exist and are equal to each other. That is,

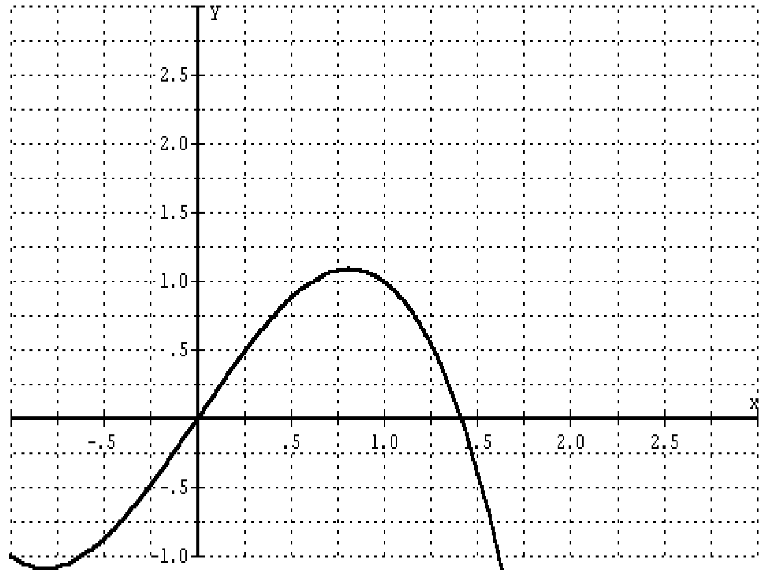
$$\begin{aligned}\lim_{n\to\infty} s(n) &= \lim_{n\to\infty}\sum_{i=1}^{n} f(m_i)\Delta x \\ &= \lim_{n\to\infty}\sum_{i=1}^{n} f(M_i)\Delta x \\ &= \lim_{n\to\infty} S(n)\end{aligned}$$

where $\Delta x = (b-a)/n$ and $f(m_i)$ and $f(M_i)$ are the minimum and maximum values of f on the subinterval.

Definition: So the Area $= \lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\Delta x, \quad x_{i-1} \le c_i \le x_i$

where $\Delta x = (b-a)/n$.

12. Find the area of the region bounded by $y = 2x - x^3$ on the interval $[0, 1]$. Use the limit definition of area.



13. Find the area of the region bounded by $y = x^2$ on the interval $[-2, 0]$. Use the limit definition of area.

14. Approximate the area of the region bounded by the curve $y = x^2 + 4x$ on the interval $[0, 4]$ using midpoints and 4 subintervals.