

## The Chain Rule

We know how to differentiate:

$$y = x^2 + 1$$

But we don't know how to differentiate:

$$y = (1 + x^2)^3$$

$$y = (1 + x^2)^{59}$$

$$y = \sqrt{1 + x^2}$$

$$y = \sin(x^2 + 1)$$

If the power is small enough, we could expand out the function and then take the derivative, but this gets time consuming for large powers. Instead we use the Chain Rule.

## The Chain Rule - The Extended Power Rule

Examples:

Differentiate :  $y = (1 + x^2)^3$

$$y = (x^2 + 1)^{59}$$

$$y = \sqrt{1 + x^2}$$

$$y = (x + 1)^3(3 - x)^4$$

$$y = \sqrt[3]{\frac{x+1}{x-2}}$$

$$y = \frac{-2}{(4x-3)^3}$$

**The Chain Rule:****Alternative Form of the Chain Rule:**

Example: Let  $y = \sqrt{u}$  and  $u = 1 - x^2$ . Find  $\frac{dy}{dx}$ .

Differentiate the following:

$$y = \sin(x^2 + 1)$$

$$y = \sin \pi x$$

$$y = \sec x^2$$

$$y = \sin^2 3x$$

$$y = \sin(\cos x)$$

$$y = \sin(\cos^2 x)$$

Theorem: Derivative of the Exponential Function with base e

Let  $u$  be a differentiable function of  $x$ .

$$1. \frac{d}{dx}[e^x] = e^x$$

$$2. \frac{d}{dx}[e^u] = e^u \frac{du}{dx} = e^u u'$$

Find the derivative of the following:

$$\frac{d}{dx}[e^{x^2-3x+1}]$$

$$\frac{d}{dx}[2e^{\frac{1}{x}}]$$

$$\frac{d}{dx}\left[\frac{1}{e^x}\right]$$

$$\frac{d}{dx}[e^{e^x}]$$

**Theorem: Derivative of the Natural Logarithmic Function**

Let  $u$  be a differentiable function of  $x$ .

$$1. \frac{d}{dx}[\ln x] = \frac{1}{x}, \quad x > 0$$

$$2. \frac{d}{dx}[\ln u] = \frac{1}{u} \frac{du}{dx} = \frac{u'}{u}, \quad u > 0$$

**Proof:**

$$\frac{d}{dx}[\ln 5x] =$$

Differentiate  $f(x) = \ln \sqrt{2x - 1}$

$$\frac{d}{dx} \left( \ln \frac{2x(x+1)^3}{\sqrt[3]{2x^2+1}} \right) =$$

$$\frac{d}{dx} [\ln(\ln x)] =$$

The exponential function to the base  $a$ :

The logarithmic function to the base  $a$ :

Derivatives for Bases Other than  $e$ :

Let  $a$  be a positive real number ( $a \neq 1$ ) and let  $u$  be a differentiable function of  $x$ .

1.  $\frac{d}{dx}[a^x] = (\ln a)a^x$

2.  $\frac{d}{dx}[a^u] = (\ln a)a^u \frac{du}{dx}$

3.  $\frac{d}{dx}[\log_a x] = \frac{1}{(\ln a)x}$

4.  $\frac{d}{dx}[\log_a u] = \frac{1}{(\ln a)u} \frac{du}{dx}$

Proof:

Differentiate the following functions:

$y = 3^x$

$y = \log_5 x^2$

$y = 2^{e^x}$

$y = \log_3(\cos 2x)$

**Challenge question**

Find the derivative of:

$$\ln \sqrt{\ln(\ln(\cos(x^3 - 2x)))}$$