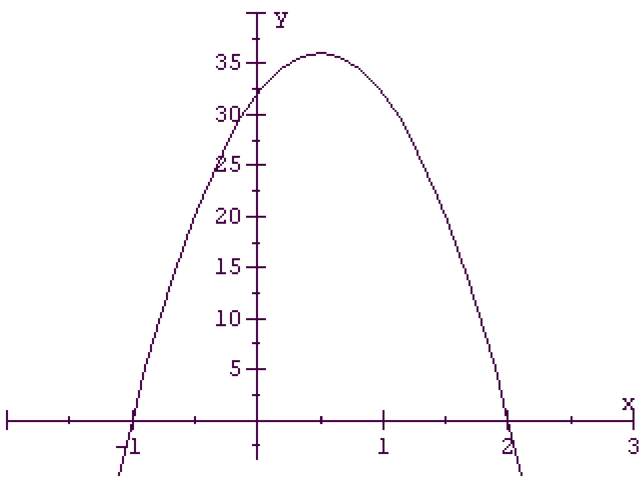
The Derivative and the Tangent Line Problem

Meaning of a tangent line to a circle at a point P:

What about a tangent line to a general curve at a point P:

We will think of a tangent line to a general curve in a way similar to a tangent line to a circle, however it should be clear that the definition will have to be modified.

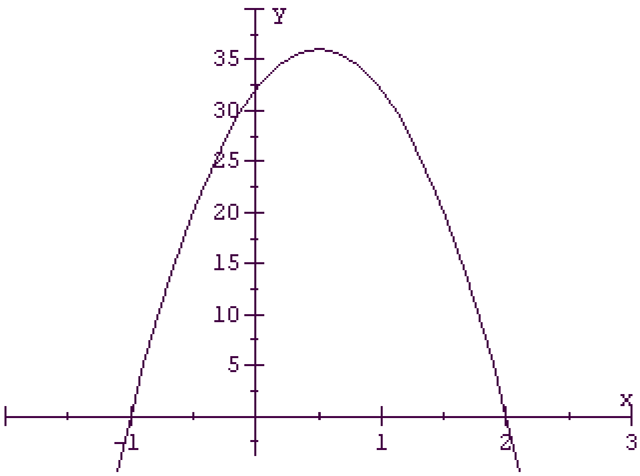
Consider the curve $f(x) = -16x^2 + 16x + 32$ with graph shown below:



Illustrate what we believe to be a tangent line to the curve at the point $(1, 32)$. One way to think about a line is in terms of its slope. What is needed to find a slope of a line?

Since we do not have this information, the best we can do is estimate the slope. How could we do this?

Let's explore an efficient way of calculating these slopes.



Let P be the point $(1, 32)$ and let Q be a point on $f(x)$. The coordinates of point Q will be $(x, f(x))$. Then the slope of the **secant line** between P and Q is:

$m =$

Find secant line slopes if point Q has x-coordinate: $f(x) = -16x^2 + 16x + 32$

x	$f(x)$	m_{PQ}
0		
$\frac{1}{2}$		
$1\frac{1}{2}$		
2		

If we want point Q to be close to the original point P (for a better approximation), then we can think of the coordinates of point Q as $(1 + \Delta x,\ f(1 + \Delta x))$. The smaller the value of Δx, the closer Q is to P.

Calculate new secant slopes for the following value of Δx:

Δx	$f(1 + \Delta x)$	m_{PQ}
.1		
.01		
.001		
.0001		

Instead of using specific numbers for point Q, let's find the slope in general (symbolically). The new slope expression between P and Q is given by:

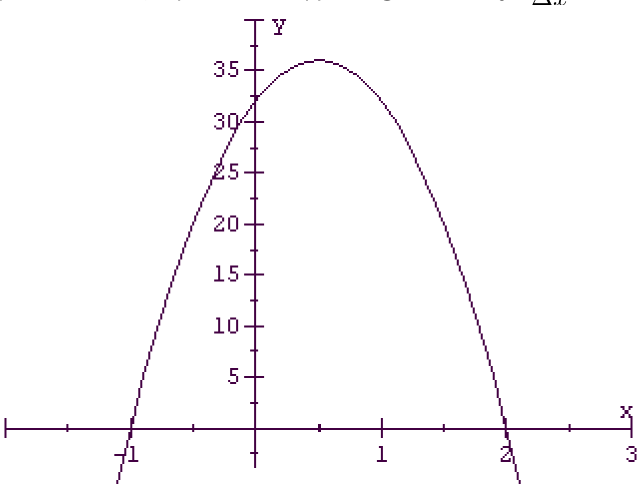
$$m = \frac{f(1+\Delta x) - f(1)}{(1+\Delta x) - 1}$$

What we are estimating through these calculations

is: $\lim\limits_{\to} \quad \frac{f(1+\Delta x)-f(1)}{\Delta x}$

This limit is the number that we assign to the slope of the tangent line to this curve at the point $(1, 32)$.

In general the slope of a **secant line** from a point $(c\,, f(c))$ to a point $(c+\Delta x,\, f\,(c+\Delta x\,))$ is given by $\frac{\Delta y}{\Delta x} = \frac{f(c+\Delta x)-f\,(c\,)}{\Delta x}$.



The result of this calculation is also called the **average rate of change of y with respect to x.** In this example the slope of each secant line that we calculated was an estimate of the slope of the tangent line at the point $(c, f\,(\,c\,))$. The actual slope is obtained by finding the limit as Δx approaches 0 in this expression.

Definition: Tangent line with slope m.

If f is defined on an open interval containing c, and if

$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(c+\Delta x) - f(c)}{\Delta x} = m$ (that is, the limit exists and has value

m), then the line passing through the point $(c, f(c))$ with slope m is the tangent line to the graph of f at the point $(c, f(c))$. Note: the **slope of the tangent line** at $(c, f(c))$ is also called the **slope of the graph** of f at $(c, f(c))$

$\lim_{\Delta x \to 0} \frac{f(c+\Delta x) - f(c)}{\Delta x}$ is called the **instantaneous rate of change** of y with respect to x.

Contrast with average rate of change: $\frac{f(c+\Delta x) - f(c)}{\Delta x}$

A note regarding the slope formula:

$m = \lim_{\Delta x \to 0} \frac{f(c+\Delta x) - f(c)}{\Delta x}$. The point $(c, f(c))$ is the point at which you are trying to find a tangent line. In our example this point was $(1, 32)$ so $c = 1$ and $f(c) = 32$.

An alternative is to think of the formula as $m = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$ where we generally find a slope formula for a given curve which can be used to find the slope at **any** point x on the curve, rather than at a specific point c. This alternative is equally easy to use and is probably a bit more useful. We just simplify this expression for the given function, then substitute in the desired point, **after** simplification

Find the equation of the tangent line to curve $f(x) = x^3 + x^2$ at the point $(1,2)$. We will do this problem using both the specific and the more general slope formula. After finding the equation, sketch the graph of the curve with the tangent line on your calculator for visualization.

$m = \lim_{\Delta x \to 0} \frac{f(c+\Delta x) - f(c)}{\Delta x}$

$m = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$

Steps for finding a Tangent Line to a curve:

1) Write down the difference quotient: $\frac{f(x+\Delta x) - f(x)}{\Delta x}$
2) Simplify the difference quotient
3) Take the limit as $\Delta x \to 0$ (this is the slope m of the tangent line)
4) Use point-slope form to find the equation of the tangent line : $y - y_1 = m(x - x_1)$

Find the equation of the tangent line to the curve $f\,(x\,) = \dfrac{1}{\sqrt{x}}$ at the point $\left(4,\ \dfrac{1}{2}\right)$.

$$m = \lim_{\Delta x \to 0} \ \frac{f(x+\Delta x) - f\,(x\,)}{\Delta x}$$

The limit used to define the slope of a tangent line is also used to define one of the two fundamental operations of calculus - **differentiation**.

Definition of the Derivative:

The **derivative** of f at x is given by

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$ (f' is pronounced f prime)

provided the limit exists. For all x for which this limit exisits, f' is a function of x.

This means that the derivative of $f(x)$ gives the slope of the tangent line to the graph of f at the point $(x, f(x))$, provided there is a tangent line at this point.

The process of finding the derivative of a function is called **differentiation**. A function is **differentiable** at x if its derivative exists at x and is **differentiable on the open interval (a, b)** if it is differentiable at every point on the interval.

Notation for the derivative of $f(x)$:

$f'(x)$ $\frac{dy}{dx}$ y' $\frac{d}{dx}[f(x)]$ $D_x[y]$

In conclusion,

$$\frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= f'(x)$$

Steps for finding the derivative:

1) Write down the difference quotient: $\frac{f(x+\Delta x)-f(x)}{\Delta x}$
2) Simplify the difference quotient
3) Take the limit as $\Delta x \to 0$

Find the derivative $f'(x)$ for each of the functions below:

$f(x) = -2x + 5$

$f(x) = 7x^2$

$f(x) = \frac{1}{1-x}$

$f(x) = \frac{2}{x}$

Find the slope of the graph of $f(x) = 3x - 2$ at the point $(3, 7)$.

Find the derivative $f'(x)$ for the following function.
Then discuss the behavior of $f(x)$ and $f'(x)$ at (0,0).

$f(x) = \sqrt{x}$

If the limit definition of the derivative does not exist at $(c, f(c))$ because it goes to $\pm\infty$, then $f(x)$ is said to have a vertical tangent line at c.

The following alternative limit form of the derivative is useful when finding the derivative at a specific point $(c,\ f(c))$:

$f'(c) = \lim\limits_{x \to c} \frac{f(x)-f(c)}{x-c}$ **(ONLY use this if you are given a specific point c!)**

Use this alternative limit form of the derivative to find $f'(0)$ for the following:

$f(x) = |x|$

$f(x) = [\![x]\!]$

$f(x) = x^{\frac{1}{3}}$

A function is **NOT DIFFERENTIABLE at a point c** if at that point there exists:

1) kinks or sharp turns 2) discontinuities 3) vertical tangent lines

Definition of a **Locally Linear function $f(x)$ at c**:

Theorem: If f is differentiable at $x = c$, then f is continuous at $x = c$.

TRUE or FALSE?

1) $y = |x - 1|$ is continuous at $x = 1$.

2) $y = |x - 1|$ is differentiable at $x = 1$.

3) If a graph is discontinuous at c, then it is not differentiable at c.

4) If a graph is continuous at c, then it is differentiable at c.

5) If a graph is differentiable at c, then it is continuous at c.

_______________ at c IMPLIES ________________ at c.

_______________ at c DOES NOT NECESSARILY IMPLY ________________ at c.

Summary of Derivative Formulas:

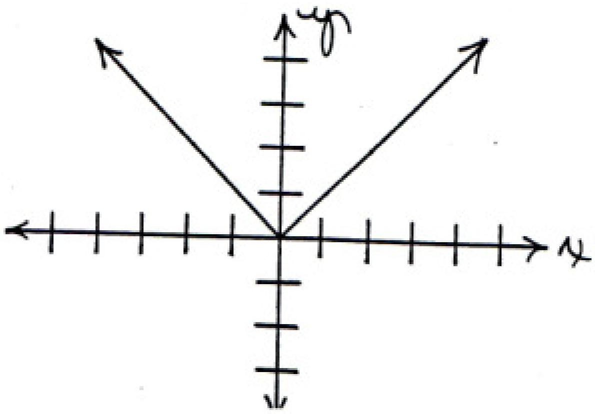
If you want $f'(x)$, then use: (for finding the derivative at any point x)
$f'(x) = \lim\limits_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$

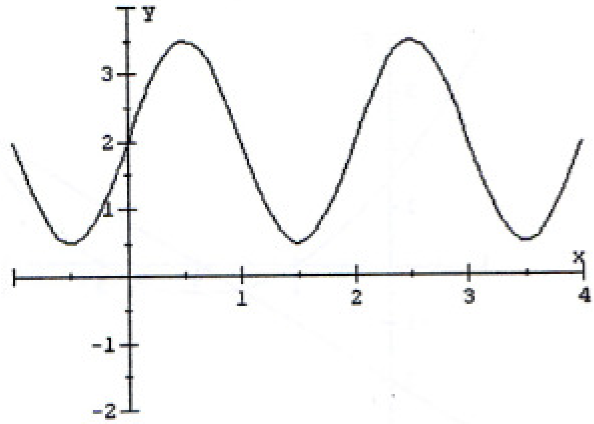
If you want $f'(c)$, then use: (for finding the derivative at a specific point c)
$f'(c) = \lim\limits_{x \to c} \frac{f(x)-f(c)}{x-c}$

To find the equation of a tangent line, you can use either form, but I would recommend using $f'(c)$. You will also usually need to find the y-value of the tangent point, $f(c)$.
Steps for finding the tangent line at $x = c$:

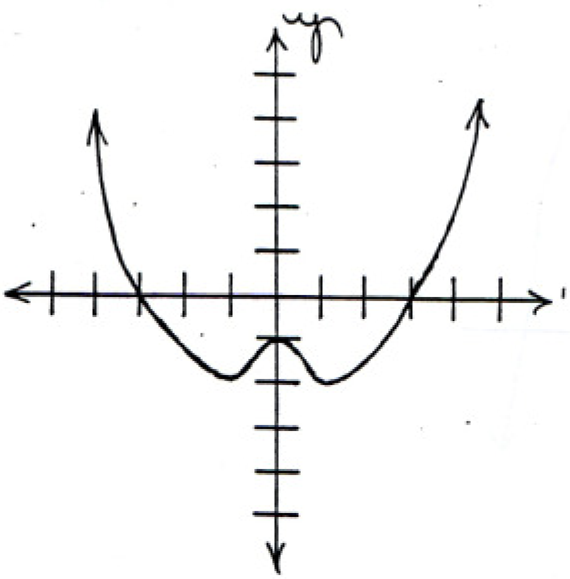
1) Calculate $f'(c)$ (this will equal the tangent slope m)
2) Find $f(c)$ (this will equal the y-value of the tangent point)
3) Plug in to point-slope form: $y - f(c) = m(x - c)$

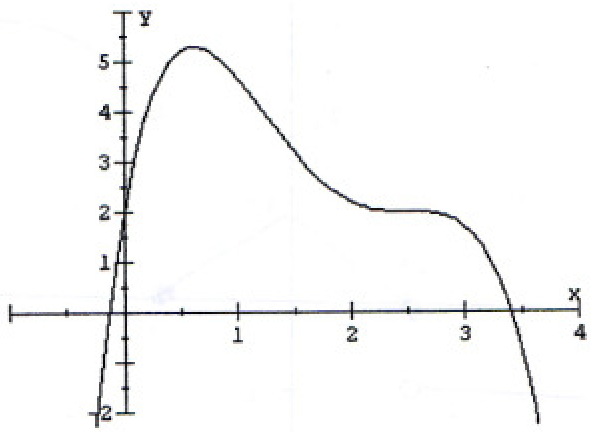
Given the graph of $f(x)$, sketch the graph of $f'(x)$:



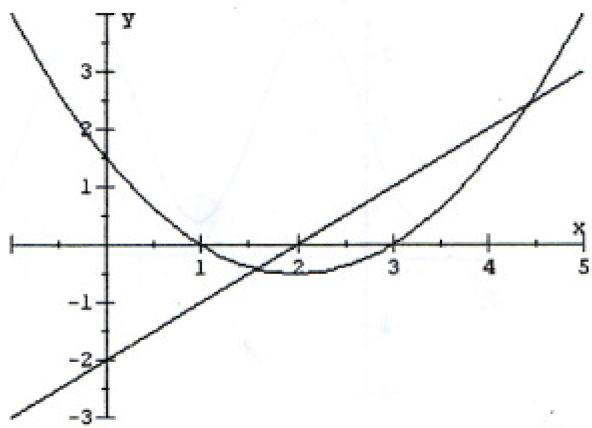


Given the graph of $f(x)$, sketch the graph of $f'(x)$ and $f''(x)$:





Which graph is $f(x)$ and which graph is $f'(x)$?



Find the graph of $f'(x)$.