Alternating Series Test

Recall that the Integral Test, Direct Comparison Test, and Limit Comparison Test all require that the terms of the series are positive. Many series such as $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$, $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$, and $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n+1}$ do not have all positive terms and thus cannot be investigated using the above mentioned tests.

We will first investigate series whose terms alternate in sign.

Theorem: Alternating Series Test (AST)

Assume $a_n > 0$. The alternating series $\sum_{n=1}^{\infty} (-1)^n a_n$ and $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converge if the following two conditions are met:

1) $a_{n+1} \leq a_n \, \forall \, n$ and 2) $\lim_{n \to \infty} a_n = 0$

1. $\sum_{n=1}^{\infty} (-1)^n \frac{1}{\ln n}$

2. $$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{n^2+1}$$

3. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^3}{n^3+6}$

4. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n}\frac{1}{n!}$

5. $\displaystyle\sum_{n=1}^{\infty} \cos n\pi$

6. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n} \cos n\pi$

There is nothing in the statement of AST that refers to what the series converges to. However there is a theorem that deals with this issue.

Theorem: Alternating Series Remainder
If a convergent alternating series satisfies the condition that $a_{n+1} \leq a_n \, \forall \, n$, then the absolute value of the remainder R_N involved in approximating the sum S by S_N is less than or equal to the first neglected term. That is $|S - S_N| \leq a_{n+1}$

Proof:

Since $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$ is convergent, the deletion of the first n terms will also result in a convergent series. Therefore, $R_N = S - S_N$ is a convergent series.

$$
\begin{aligned}
R_N &= S - S_N \\
&= \sum_{n=N+1}^{\infty} (-1)^{n-1} a_n \\
&= (-1)^N a_{N+1} + (-1)^{N+1} a_{N+2} + (-1)^{N+2} a_{N+3} + \ \ldots \\
&= (-1)^N (a_{N+1} - a_{N+2} + a_{N+3} - a_{N+4} + a_{N+5} \ldots) \\
|R_N| &= (a_{N+1} - a_{N+2} + a_{N+3} - a_{N+4} + a_{N+5} \ldots) \\
&= a_{N+1} - (a_{N+2} - a_{N+3}) - (a_{N+4} - a_{N+5}) - (a_{N+6} - a_{N+7}) \ldots
\end{aligned}
$$

Since $a_{N+1} \leq a_N \, \forall N$,

$$|R_N| \leq a_{N+1}.$$

7. Approximate the sum of the convergent series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{(2n)!}$ with an error $< .001.$

8. Find the number of terms to include in the sum of the series $\sum_{n=1}^{\infty}(-1)^{n}\frac{1}{n}\left(\frac{1}{2}\right)^{n}$ so $R_N < .0005$

Definition: A series $\sum_{n=1}^{\infty} a_n$ is <u>absolutely convergent</u> if the series $\sum_{n=1}^{\infty} |a_n|$ is convergent. Note: If $\sum_{n=1}^{\infty} a_n$ has all positive terms, $\sum_{n=1}^{\infty} a_n$ is the same as $\sum_{n=1}^{\infty} |a_n|$. Therefore, all convergent series having only positive terms are absolutely convergent.

Definition: A series $\sum_{n=1}^{\infty} a_n$ is <u>conditionally convergent</u> if $\sum_{n=1}^{\infty} a_n$ is convergent and $\sum_{n=1}^{\infty} |a_n|$ is divergent.

Determine whether each of the following is absolutely convergent (AC) or conditionally convergent (CC) or divergent.

9. $\sum_{n=1}^{\infty}(-1)^n \frac{1}{n^2}$

10. $\sum_{n=1}^{\infty}(-1)^n \frac{1}{n}$

11. $\sum_{n=1}^{\infty}(-1)^n \frac{n}{n+2}$

Determine whether each of the following is absolutely convergent (AC) or conditionally convergent (CC) or divergent.

12. $\displaystyle\sum_{n=1}^{\infty}(-1)^n \frac{3^n}{2^{n+1}}$

13. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n+1} \frac{n}{5+3n^2}$

We have theorems to deal with infinite series which have all positive terms and with infinite series which have terms that are alternating positive and negative but nothing to deal with series that are randomly positive and negative such as $\sum_{n=1}^{\infty} \frac{\sin n}{n^3}$. The following theorem can be used in situations such as this.

Theorem: Absolute Convergence Theorem: (ACT)

If $\sum_{n=1}^{\infty} |a_n|$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

Proof:

Assume $\sum_{n=1}^{\infty} |a_n|$ converges and try to prove $\sum_{n=1}^{\infty} a_n$ converges.

We will first show that $\sum_{n=1}^{\infty} (a_n + |a_n|)$ converges.

$0 \le a_n + |a_n| \le 2|a_n| \; \forall \; n$

Therefore $\sum_{n=1}^{\infty} (a_n + |a_n|)$ converges by CT with $\sum_{n=1}^{\infty} 2|a_n|$ which converges because it is a constant times $\sum_{n=1}^{\infty} |a_n|$ which is a given convergent series.

$$\begin{aligned}
\text{Also } \sum_{n=1}^{\infty} a_n &= \sum_{n=1}^{\infty} (a_n + |a_n| - |a_n|) \\
&= \sum_{n=1}^{\infty} ((a_n + |a_n|) - |a_n|) \\
&= \sum_{n=1}^{\infty} (a_n + |a_n|) - \sum_{n=1}^{\infty} |a_n|
\end{aligned}$$

Sometimes you can see that is will be fairly easy to show that an absolute value series converges. If you can do this, you can conclude that the original series also converges.

14. $\displaystyle\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$

Can't use CT, LCT, IT, or AST.

Consider $\displaystyle\sum_{n=1}^{\infty} \left| \frac{\sin n}{n^2} \right| = \sum_{n=1}^{\infty} \frac{|\sin n|}{n^2}$

Since $0 \le |\sin n| \le 1 \,\forall\, n \Rightarrow \dfrac{0}{n^2} \le \dfrac{|\sin n|}{n^2} \le \dfrac{1}{n^2}$

$\therefore \displaystyle\sum_{n=1}^{\infty} \frac{|\sin n|}{n^2}$ converges using CT with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ a convergent p-series with $p = 2 > 1$.

$\therefore \displaystyle\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$ is absolutely convergent, and therefore convergent by the Absolute Convergence Theorem.

Determine whether each of the following is convergent or divergent.

15. $\sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{(2n-1)\pi}{2}$

16. $\sum_{n=1}^{\infty} \frac{1-\cos n}{n^2}$

17. $\sum_{n=1}^{\infty} \frac{\cos n}{n^3}$

What is the importance of Absolute Convergence?

- It is the only way to show a series such as $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$ converges.
- If a series is absolutely convergent with a sum of S, then any rearrangement of the terms will also converge to S.
- If a series is conditionally convergent with a sum of S, a rearrangement of terms can make the series converge to any real number, or can make the series diverge, or can make the series oscillate.

18. As an illustration of the last point consider the series $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n}$ This series is conditionally convergent and can be shown to converge to $\ln 2$.