Arc Length and Surfaces of Revolution

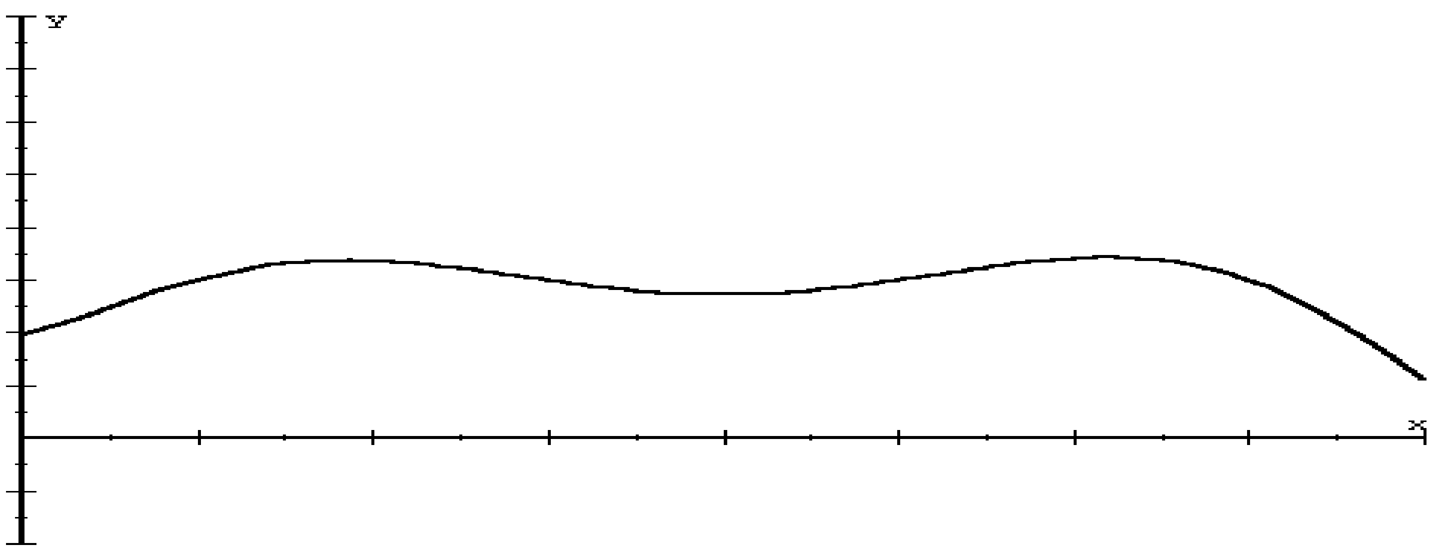
In this section we use definite integrals to find the length of an arc and to find the surface area of a solid of revolution. In order to find these, we need the Distance Formula.

Distance Formula: Given 2 points $P_1(x_{1,}y_1)$ and $P_2(x_{2,}y_2)$the distance between the points is given by the formula

$D =$.

In order to ensure that a curve has finite arc length the function representing that curve must have a continuous first derivative. Such a curve is said to be **rectifiable**. If $f'(x)$ is continuous, that will guarantee that the graph of $f(x)$ is continuous and is **smooth** with no corners or cusps.

Partition $a = x_0 < x_1 < x_2 < \ . \ . \ . \ < x_n = b$



$\Delta x_i =$ $\Delta y_i =$

$$s \approx \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2} + \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} + \sqrt{(x_3 - x_2)^2 + (y_3 - y_2)^2} +$$

$$\sqrt{(x_4 - x_3)^2 + (y_4 - y_3)^2} + \ldots + \sqrt{(x_n - x_{n-1})^2 + (y_n - y_{n-1})^2}$$

$$s \approx \sum_{i=1}^{n}$$

$$=$$

$$=$$

The **arc length** of a rectifiable function $\boldsymbol{y = f(x)}$ between (a, c) and (b, d) is given by

$$s = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx$$ or

if $x = g(y)$, the arc length is given by

$$s = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy$$

Examples:

1. Find the length of the curve given by $y = x^{\frac{2}{3}}$ from $(8, 4)$ to $(27, 9)$.

2. Find the length of the curve given by $y = \dfrac{x^3}{6} + \dfrac{1}{2x}$ on the interval $\left[\dfrac{1}{2}, 2\right]$.

3. Find the length of the curve given by $x = \dfrac{y^4}{16} + \dfrac{1}{2y^2}$ from $\left(\dfrac{9}{8}, -2\right)$ to $\left(\dfrac{9}{16}, -1\right)$.

Definition of surface area of revolution: The surface area of revolution is the surface area created when the graph of a continuous function is revolved about a line.

Area of a Surface of Revolution:

$S = 2\pi \int_a^b r(x)\sqrt{1+[f'(x)]^2}\,dx$ where $r(x)$ is the distance between the curve and the axis of revolution.

or

$S = 2\pi \int_c^d r(y)\sqrt{1+[f'(y)]^2}\,dy$ where $r(y)$ is the distance between the curve and the axis of revolution.

Either formula may be used and the formula is independent of the axis of revolution.

If a function is invertible, choose $r = x$ or y.
If not invertible, then use whatever variable the function is in terms of.

4. Find the surface area of revolution formed by revolving the graph of $y = x^3$ from $(1, 1)$ to $(2, 8)$ about the x-axis.

5. Find the area of the surface formed by revolving the graph of $f(x) = x^2$ on the interval $[0, \sqrt{2}]$ about the y-axis.

6. Derive the formula for the surface area of a cone.