Comparison Tests

Summary to Date:

Convergence

Geometric Series: $\sum ar^n$ converges if $0 < |r| < 1$

P-Series: $\sum \frac{1}{n^p}$ converges if $p > 1$

Integral Test: $\sum a_n$ converges if $\int_1^\infty f(x)dx$ converges

(need f positive, continuous and decreasing)

Partial Sums: If $\lim\limits_{n\to\infty} S_n = S$ converges, then $\sum a_n$ converges to S

Divergence

Geometric Series: $\sum ar^n$ diverges if $|r| \geq 1$

P-Series: $\sum \frac{1}{n^p}$ diverges if $0 < p \leq 1$

Integral Test: $\sum a_n$ diverges if $\int_1^\infty f(x)dx$ diverges

(need f positive, continuous and decreasing)

Partial Sums: If $\lim\limits_{n\to\infty} S_n$ diverges, then $\sum a_n$ diverges

nth Term Test: If $\lim\limits_{n\to\infty} a_n \neq 0$, then $\sum a_n$ diverges

Note: If $\lim\limits_{n\to\infty} a_n = 0$, then you can not tell convergence or divergence.

Theorem: Direct Comparison Test (CT)
Let $0 < a_n \leq b_n \, \forall \, n$ Note: This implies that all terms are positive. However if the terms are all positive after a particular term then the Direct Comparison test can still be applied. There are two possible conclusions using CT.

1. If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

If the larger series converges, then the smaller series must converge.

2. If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

If the smaller series diverges, then the larger series must diverge.

Proof:

1) Assume $\sum b_n$ converges, let $L = \sum_{n=1}^{\infty} b_n$ and let $S_n = a_1 + a_2 + ... + a_n$

Because $0 < a_n \leq b_n$, the sequence $S_1, S_2, S_3, ...$ is

2) Assume $\sum a_n$ diverges. Let $S_n = a_1 + a_2 + ... + a_n$ and $S_n = b_1 + b_2 + ... + b_n$

Because $0 < a_n \leq b_n \; \forall n$, then

Conclusive comparisons can be made between fractions with either like numerators or like denominators. We must compare the terms of an unknown series to the terms of a "known series".

1. $$\sum_{n=1}^{\infty} \frac{1}{n^3+1}$$

2. $$\sum_{n=2}^{\infty} \frac{1}{\sqrt{n-1}}$$

3. $$\sum_{n=1}^{\infty} \frac{2^n}{3^n+5}$$

4. $$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+1}}$$

5. $\displaystyle\sum_{n=1}^{\infty} \frac{|\sec n|}{n}$

6. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2+\sqrt{n}}$

Often times a given series closely resembles a geometric series or a p-series, but you cannot establish the term by term comparison necessary to apply the Direct Comparison Test. A useful test in these circumstances is the Limit Comparison Test.

Theorem: Limit Comparison Test (LCT)
Suppose $a_n > 0$ and $b_n > 0$ (positive terms) and $\lim_{n\to\infty} \frac{a_n}{b_n} = L > 0$. Then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ either both converge or both diverge. Note: In using the LCT $\sum_{n=1}^{\infty} a_n$ is generally the series to be tested and $\sum_{n=1}^{\infty} b_n$ is the series that you already know to be convergent or divergent.

Addendum to LCT:

1. If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$ and $\sum_{n=1}^{\infty} b_n$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is also convergent.

2. If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$ and $\sum_{n=1}^{\infty} b_n$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is also divergent.

1. If $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$, then b_n overpowers a_n. (b_n is the bigger series so in the idea of the CT, if the bigger series b_n converges, so must the smaller series a_n).

2. If $\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$, then a_n overpowers b_n. (a_n is the bigger series, so if b_n diverges, so does the bigger series a_n.)

7. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{an+b}$ (general harmonic) $a > 0,\ b > 0$

8. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n - 5}$

9. $$\sum_{n=1}^{\infty} \frac{5n-3}{n^2-2n+5}$$

10. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{(n+1)2^{n-1}}$

11. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n + \sqrt{n^2+1}}$

Note: In using either the CT or the LCT we are able to determine whether a series converges or diverges but if it converges we do not know what it converges to, nor do we have a way to approximate the sum of the series.