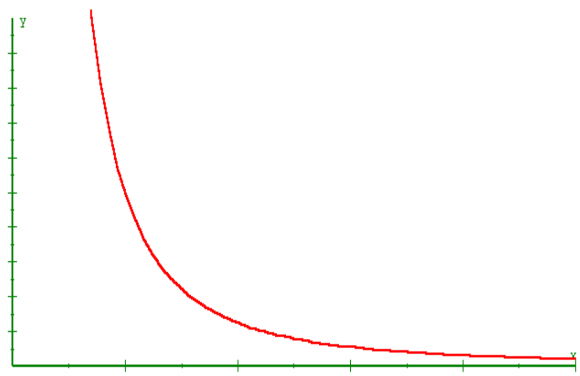
Improper Integrals

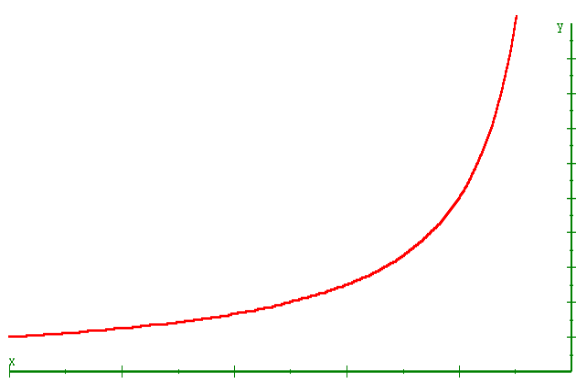
The definition of a definite integral $\int_a^b f(x)\,dx$ requires that the interval $[a,b]$ be finite. The Fundamental Theorem of Calculus requires that the function $f(x)$ is continuous on the interval $[a,b]$. In this section we will learn how to evaluate integrals if these conditions are not met. i.e. one or both of the limits of integration are infinite or the function has a finite number of infinite discontinuities in the interval $[a,b]$. An integral of this type is said to be an improper integral.

Examples of integrals that have infinite integration limits would include the following:

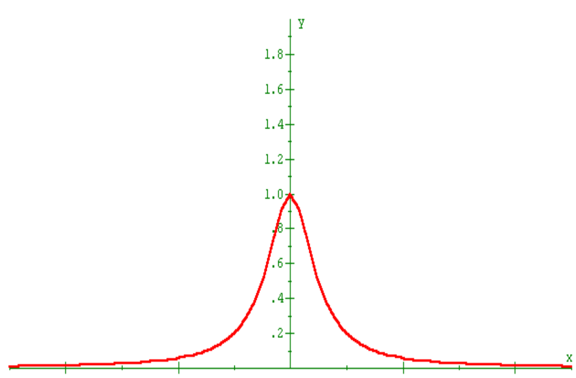
1. $\int_1^\infty \frac{1}{x^2}\,dx$



2. $\int_{-\infty}^{-1} \frac{1}{x}\,dx$



3. $\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx$



To get an idea of how to evaluate an improper integral with an infinite limit such as $\int_1^{\infty} \frac{1}{x^2}\,dx$, consider the integral $\int_1^{b} \frac{1}{x^2}\,dx$. This integral can be interpreted as the area of the region between

The value of the integral $\int_1^{b} \frac{1}{x^2}\,dx$ is

As $b \to \infty$, the value of the integral is

This improper integral can be interpreted as the unbounded area between

Definition of Improper Integrals with Infinite Integration Limits.

1. If $f(x)$ is continuous on the interval $[a, \infty)$, then

$$\int_a^{\infty} f(x)\,dx = \lim_{b\to\infty} \int_a^{b} f(x)\,dx.$$

2. If $f(x)$ is continuous on the interval $(-\infty, b]$, then

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{a\to-\infty} \int_a^{b} f(x)\,dx.$$

3. If $f(x)$ is continuous on the interval $(-\infty, \infty)$, then

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{c} f(x)\,dx + \int_c^{\infty} f(x)\,dx.$$

In the first two cases, the improper integral **converges** if the limit exists, otherwise, the improper integral **diverges.** In the third case, the improper integral on the left diverges if either of the improper integrals on the right diverges.

Examples:

1. $\displaystyle\int_1^{\infty} \frac{1}{x}\,dx$ Think about the integral as an area.

2. $\displaystyle\int_{-\infty}^{-1} \frac{1}{x^2}\,dx$

3. $$\int_{-\infty}^{\infty} \frac{1}{x^2+1}\,dx$$

4. $$\int_{-\infty}^{0} xe^{2x}\,dx$$

5. $\displaystyle\int_{-\infty}^{0} xe^{-2x}\,dx$

6. $\displaystyle\int_{0}^{\infty} \sin x\,dx$

Definition of Improper Integrals with Infinite Discontinuities.

1. If $f(x)$ is continuous on the interval $[a, b)$ and has an infinite discontinuity at b, then $\int_a^b f(x)\,dx = \lim_{c \to b^-} \int_a^c f(x)\,dx.$

2. If $f(x)$ is continuous on the interval $(a, b]$ and has an infinite discontinuity at a, then $\int_a^b f(x)\,dx = \lim_{c \to a^+} \int_c^b f(x)\,dx.$

3. If $f(x)$ is continuous on the interval $[a, b]$, except for some c in (a, b) at which $f(x)$ has an infinite discontinuity, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

In the first two cases, the improper integral **converges** if the limit exists, otherwise, the improper integral **diverges.** In the third case, the improper integral on the left diverges if either of the improper integrals on the right diverges.

7. $\int_0^9 \frac{1}{\sqrt{x}}\,dx$

8. $$\int_{-1}^{0} \frac{1}{\sqrt[3]{x+1}}\,dx$$

9. $$\int_{-1}^{1} x^{-\frac{4}{3}}\,dx$$

10. $\displaystyle\int_{-\infty}^{0} \frac{1}{x+2}\,dx$

Comparison Test for Integrals:

If $0 \leq f(x) \leq g(x)$ on $[a, \infty)$ and if $\displaystyle\int_{a}^{\infty} g(x)\,dx$ converges, then $\displaystyle\int_{a}^{\infty} f(x)\,dx$ also converges. If $\displaystyle\int_{a}^{\infty} f(x)\,dx$ diverges, then $\displaystyle\int_{a}^{\infty} g(x)\,dx$ also diverges.

11. Use the comparison test to determine the divergence or convergence of the integral: $\int_2^\infty \frac{2\cos^2 x}{x^2+4}\,dx$

12. Use the comparison test to determine the divergence or convergence of the integral: $\int_1^\infty \frac{x^2+3x}{x^3-4x+5}\,dx$

13. Find the volume when the region bounded by the curve $f(x) = \dfrac{1}{x}$, $x \geq 1$ and the x-axis is revolved about the x-axis.

14. Find the surface area of the volume formed in problem number 13.

15. Prove: $\displaystyle\int_1^\infty \frac{dx}{x^p} = \begin{cases} \frac{1}{p-1} & \text{if } p > 1 \\ \text{diverges} & \text{if } p \leq 1 \end{cases}$