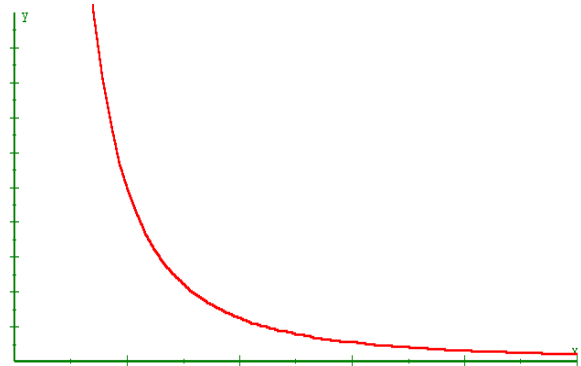


## Improper Integrals

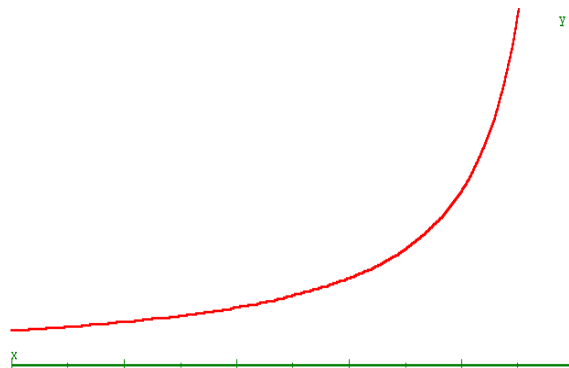
The definition of a definite integral  $\int_a^b f(x) dx$  requires that the interval  $[a, b]$  be finite. The Fundamental Theorem of Calculus requires that the function  $f(x)$  is continuous on the interval  $[a, b]$ . In this section we will learn how to evaluate integrals if these conditions are not met. i.e. one or both of the limits of integration are infinite or the function has a finite number of infinite discontinuities in the interval  $[a, b]$ . An integral of this type is said to be an improper integral.

Examples of integrals that have infinite integration limits would include the following:

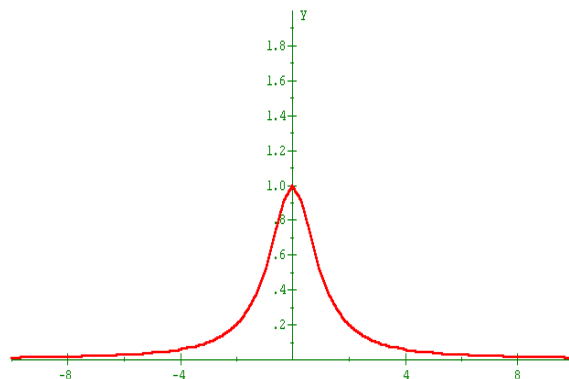
1. 
$$\int_1^{\infty} \frac{1}{x^2} dx$$



2. 
$$\int_{-\infty}^{-1} \frac{1}{x} dx$$



3. 
$$\int_{-\infty}^{\infty} \frac{1}{x^2 + 1} dx$$



To get an idea of how to evaluate an improper integral with an infinite limit such as  $\int_1^{\infty} \frac{1}{x^2} dx$ , consider the integral  $\int_1^b \frac{1}{x^2} dx$ . This integral can be interpreted as the area of the region between

The value of the integral  $\int_1^b \frac{1}{x^2} dx$  is

As  $b \rightarrow \infty$ , the value of the integral is

This improper integral can be interpreted as the unbounded area between

### Definition of Improper Integrals with Infinite Integration Limits.

1. If  $f(x)$  is continuous on the interval  $[a, \infty)$ , then

$$\int_a^{\infty} f(x) dx = \lim_{b \rightarrow \infty} \int_a^b f(x) dx.$$

2. If  $f(x)$  is continuous on the interval  $(-\infty, b]$ , then

$$\int_{-\infty}^b f(x) dx = \lim_{a \rightarrow -\infty} \int_a^b f(x) dx.$$

3. If  $f(x)$  is continuous on the interval  $(-\infty, \infty)$ , then

$$\int_{-\infty}^{\infty} f(x) dx = \int_{-\infty}^c f(x) dx + \int_c^{\infty} f(x) dx.$$

In the first two cases, the improper integral **converges** if the limit exists, otherwise, the improper integral **diverges**. In the third case, the improper integral on the left diverges if either of the improper integrals on the right diverges.

Examples:

1.  $\int_1^{\infty} \frac{1}{x} dx$

Think about the integral as an area.

2.  $\int_{-\infty}^{-1} \frac{1}{x^2} dx$

**3.**  $\int_{-\infty}^{\infty} \frac{1}{x^2 + 1} dx$

4.  $\int_{-\infty}^0 x e^{2x} dx$

**5.**  $\int_{-\infty}^0 x e^{-2x} dx$

**6.**  $\int_0^{\infty} \sin x dx$

## Definition of Improper Integrals with Infinite Discontinuities.

1. If  $f(x)$  is continuous on the interval  $[a, b)$  and has an infinite

discontinuity at  $b$ , then 
$$\int_a^b f(x) dx = \lim_{c \rightarrow b^-} \int_a^c f(x) dx.$$

2. If  $f(x)$  is continuous on the interval  $(a, b]$  and has an infinite

discontinuity at  $a$ , then 
$$\int_a^b f(x) dx = \lim_{c \rightarrow a^+} \int_c^b f(x) dx.$$

3. If  $f(x)$  is continuous on the interval  $[a, b]$ , except for some  $c$  in  $(a, b)$  at which  $f(x)$  has an infinite discontinuity, then

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx.$$

In the first two cases, the improper integral **converges** if the limit exists, otherwise, the improper integral **diverges**. In the third case, the improper integral on the left diverges if either of the improper integrals on the right diverges.

7. 
$$\int_0^9 \frac{1}{\sqrt{x}} dx$$

**8.**  $\int_{-1}^0 \frac{1}{\sqrt[3]{x+1}} dx$

**9.**  $\int_{-1}^1 x^{-\frac{4}{3}} dx$



10.  $\int_{-\infty}^0 \frac{1}{x+2} dx$

**Comparison Test for Integrals:**

If  $0 \leq f(x) \leq g(x)$  on  $[a, \infty)$  and if  $\int_a^{\infty} g(x) dx$  converges, then  $\int_a^{\infty} f(x) dx$  also converges. If  $\int_a^{\infty} f(x) dx$  diverges, then  $\int_a^{\infty} g(x) dx$  also diverges.

11. Use the comparison test to determine the divergence or convergence of the integral:  $\int_2^{\infty} \frac{2\cos^2 x}{x^2+4} dx$

- 12.** Use the comparison test to determine the divergence or convergence of the integral:  $\int_1^{\infty} \frac{x^2+3x}{x^3-4x+5} dx$

**13.** Find the volume when the region bounded by the curve  $f(x) = \frac{1}{x}$   
 $x \geq 1$  and the  $x$ -axis is revolved about the  $x$ -axis.

**14.** Find the surface area of the volume formed in problem number 13.

**15. Prove:**  $\int_1^{\infty} \frac{dx}{x^p} = \begin{cases} \frac{1}{p-1} & \text{if } p > 1 \\ \text{diverges} & \text{if } p \leq 1 \end{cases}$