Integral Test and P-Series

Consider the improper integral results:

$$\int_1^{\infty} \frac{1}{x} dx$$

compared with

$$\int_1^{\infty} \frac{1}{x^2}\, dx$$

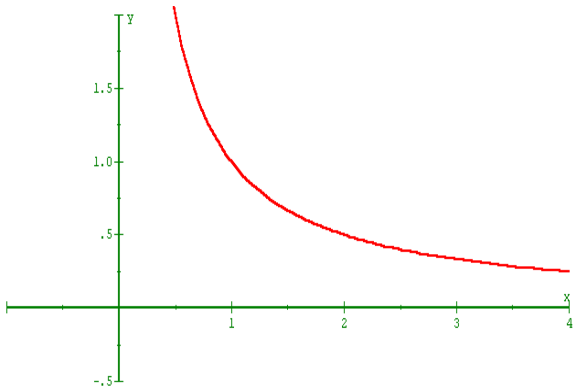
What does this imply about:

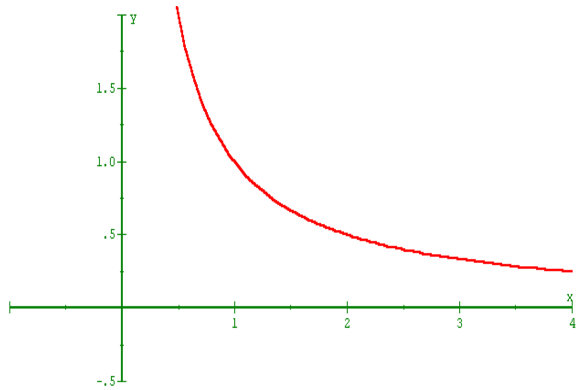
$$\sum_{n=1}^{\infty} \frac{1}{n}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

Integral Test: If f is positive, continuous, and decreasing $\forall\, x \geq 1$ and $a_n = f(n)$ (n a positive integer), then $\sum_{n=1}^{\infty} a_n$ and $\int_1^{\infty} f(x)\,dx$ either both converge or both diverge.

Proof:





Area of inscribed rectangles:

Area of circumscribed rectangles

Let $S_n = f(1) + f(2) + ... + f(n)$, then

$\sum_{i=2}^{n} f(i) =$ and $\sum_{i=1}^{n-1} f(i) =$

The exact area $\int_1^n f(x)dx$ is between the inscribed and circumscribes area, so

Proof: Assume $\int_1^{\infty} f(x)dx$ converges, then for $n \geq 1$

Proof: Assume $\int_1^{\infty} f(x)dx$ diverges, then

Examples:

1. $\sum_{n=1}^{\infty} n e^{-n}$

Note: $\lim_{x \to \infty} x e^{-x} =$

2. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^3}$ Note: $\displaystyle\lim_{x\to\infty} \frac{1}{x(\ln x)^3} =$

3. $\displaystyle\sum_{n=1}^{\infty} \ln n$

Note: $\displaystyle\lim_{x \to \infty} \ln x =$

4. $\displaystyle\sum_{n=1}^{\infty} \frac{\ln n}{n}$

Theorem: For any positive integer k, the series $\displaystyle\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \ldots$ and the series $\displaystyle\sum_{n=k}^{\infty} a_n = a_k + a_{k+1} + \ldots$ either both converge or both diverge. That is - the first k terms do not affect the convergence/divergence of an infinite series.

Theorem on Convergence/Divergence of the p-series

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \ldots$ **(called the p-series) converges if $p > 1$ and diverges if $p \leq 1$.**

Proof:

Since $f(x) = \dfrac{1}{x^p}$ is continuous, positive and decreasing $f'(x) = \dfrac{-p}{x^{p-1}} < 0 \, \forall \, x$ for $p \geq 1$, we can apply the integral test.

Recall that we have proved that $\displaystyle\int_1^{\infty} \frac{1}{x^p}\,dx$ (p positive) converges if $p > 1$ and diverges if $p \leq 1$. Therefore by the Integral Test the theorem is proven.

Note: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^1}$ **i.e. the p-series when $p = 1$ is called the Harmonic Series.**

Note: Even though we know whether or not a p-series converges, we do not know what it converges to!

5. $$\sum_{n=1}^{\infty} \frac{1}{n^{.95}}$$

6. $$\sum_{n=1}^{\infty} \frac{1}{n^{\frac{4}{3}}}$$

7. $$\sum_{n=1}^{\infty} \frac{1}{n^{\pi}}$$

8. $$\sum_{n=1}^{\infty} (1.075)^n$$

9. $$\sum_{n=1}^{\infty} \left(\frac{1}{n^2} - \frac{1}{n^3} \right)$$

10. Alternate proof of divergence of Harmonic Series.

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \dots$$

Recall $S_1 = 1$, $S_2 = 1.5$, $S_3 = 1.83$, $S_4 = 2.08$, $S_{100} = 5.19$, $S_{500} = 6.79$, $S_{1000} = 7.49$, $S_{2000} = 8.18$, $S_{3000} = 8.59$, $S_{248,642} = 13.0067$.

We will show that for any $N > 0$, we can make $S_n > N$

$$1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \left(\frac{1}{9} + \dots + \frac{1}{16}\right) + \left(\frac{1}{17} + \dots + \frac{1}{32}\right) + \dots$$

$S_4 >$

$S_8 >$

$S_{16} >$

$S_{32} >$

$S_{2^k} >$

$S_{2^{12}} >$

$S_{2^{18}} >$

11. How many terms of the harmonic series are needed to obtain a sum greater than 20?

12. How many terms of the harmonic series are needed to obtain a sum greater than 100?

13. How many terms of the harmonic series are needed to obtain a sum greater than 1660?

Miscellaneous Results:

1. If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are convergent series, then $\sum_{n=1}^{\infty}(a_n + b_n)$ converges and $\sum_{n=1}^{\infty} ca_n$ converges.

2. If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} ca_n$ diverges.

3. If $\sum_{n=1}^{\infty} a_n$ converges and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty}(a_n + b_n)$ diverges.

4. If $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are divergent series, then no conclusion can be drawn regarding $\sum_{n=1}^{\infty}(a_n + b_n)$.

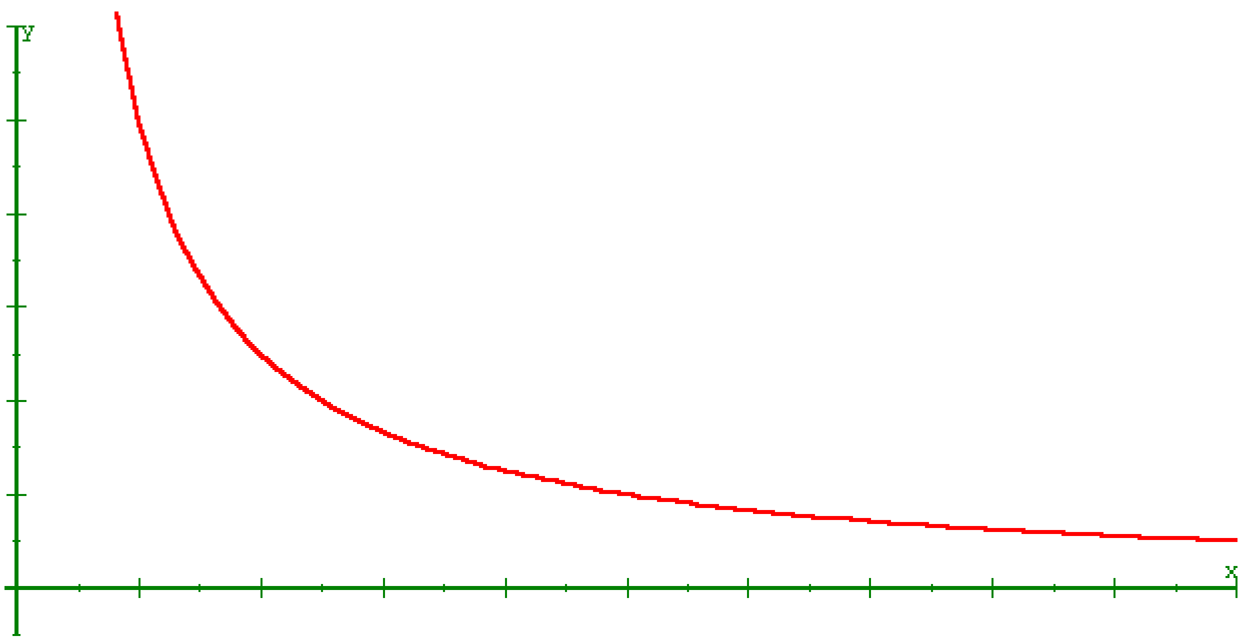
Compare $\sum_{n=1}^{\infty} \frac{1}{n} + \sum_{n=1}^{\infty} \frac{1}{2n}$ with $\sum_{n=1}^{\infty} \frac{1}{n} + \sum_{n=1}^{\infty} -\frac{1}{n}$

Although the Integral Test can be used to determine whether or not an infinite series converges or diverges it does not indicate what a convergent series converges to.

We will now investigate a method in which we can find an approximation to the sum of a convergent infinite series if the series satisfies the hypotheses of the Integral Test.

Definition: The remainder of a series is denoted by R_N and is represented by the sum of all of the terms S minus the sum of the first N terms S_N. i.e. $R_N = S - S_N$.

Consider a positive, continuous, decreasing function such as the one shown in the accompanying graph.



$$S_N = a_1 + a_2 + a_3 + \ \ldots \ + a_N$$

$$S = a_1 + a_2 + a_3 + \ \ldots \ + a_N + a_{N+1} + a_{N+2} + a_{N+3} + \ \ldots$$

$$R_N = S - S_N = a_{N+1} + a_{N+2} + a_{N+3} + \ \ldots$$

$$0 \leq R_N \leq \int_N^\infty f(x)\,dx \text{ where } f(n) = a_n$$

14. Approximate the sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^5}$ using the first 4 terms and estimate the maximum error approximation.

$$\sum_{n=1}^{4} \frac{1}{n^5} = 1 + \frac{1}{2^5} + \frac{1}{3^5} + \frac{1}{4^5} \approx 1.0363417784$$

Recall $0 \leq R_4 \leq \int_4^{\infty} \frac{1}{x^5} dx = \lim_{b \to \infty} \int_4^{b} x^{-5} dx =$

15. Determine the number of terms to include in the sum in order to find the sum of $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}}$ so that the truncation error $(R_N) \leq .001$