Moments, Center of Mass, Centroids

Mass: the measure of a body's resistance to changes in motion
(independent of gravity)

Weight: $\boldsymbol{W = mg}$

Weight = Mass x Gravity (Gravity depends on the gravitational system)
(On Earth $g = 9.8\,\frac{\text{m}}{\text{s}^2}$ or $32\,\frac{\text{ft}}{\text{s}^2}$)

Force: $\boldsymbol{F = ma}$

Force = Mass x Acceleration due to gravity

Measures of mass:

U.S. - Slug

International - Kilogram

CGS - Gram

Measures of force:

U.S. - Pound

International - Newton

CGS - Dyne

Consider a seesaw:

A moment of m about the point P:

Left moment about point P:

Right moment about point P:

To balance the seesaw, the two moments must be equal.

Change mass:

Change distance:

Consider as points on a number line:

The tendency of this system to rotate about the origin is called the **moment about the origin:**

$$M_0 = m_1x_1 + m_2x_2 + m_3x_3 + \ldots + m_nx_n$$

If $M_0 = 0$, then the system in in equilibrium.

Equilibrium will occur if $M_0 = 0$.

If equilibrium does not occur, then there will be a center of mass.

Center of mass:($\overline{x}$) the point $\overline{x}$ at which the fulcrum could be relocated to attain equilibrium.

If you translate the system $\overline{x}$ units, then each coordinate x_i becomes $(x_i - \overline{x})$.
After shifting, the system will now be in equilibrium so the Moment $M_0 = 0$.

Moments and Center of Mass: One-Dimensional System

Let the point masses $m_1, m_2, m_3, \ldots, m_n$ be located at $x_1, x_2, x_3, \ldots, x_n$

1. The **moment about the origin** is $M_0 = m_1x_1 + m_2x_2 + \ldots + m_nx_n$

2. The **center of mass** is $\overline{x} = \frac{M_0}{m}$, where $m = m_1 + m_2 + \ldots + m_n$ is the total mass of the system.

Examples:

1. Find the center of mass of the following point masses lying on the x-axis: $m_1 = 7, m_2 = 4, m_3 = 3,\ m_4 = 8;\ x_1 = -3,\ x_2 = -2,\ x_3 = 5,$ $x_4 = 6.$

In a 2-dimensional system, we define 2 moments:
one with respect to x and one with respect to y.

Moments and Center of Mass: Two-Dimensional System

Let the point masses m_1, m_2, m_3, ..., m_n be located at (x_1, y_1), (x_2, y_2), (x_3, y_3), ..., (x_n, y_n)

1. The **moment about the y-axis** is $My = m_1x_1 + m_2x_2 + \ldots + m_nx_n$

2. The **moment about the x-axis** is $Mx = m_1y_1 + m_2y_2 + \ldots + m_ny_n$

3. The **center of mass** ($\overline{x}$,$\overline{y}$) (or **center of gravity**) is

   $\overline{x} = \frac{My}{m}$ and $\overline{y} = \frac{Mx}{m}$

   where $m = m_1 + m_2 + \ldots + m_n$ is the total mass of the system.

2. Find the center of mass of a system having masses $2, 3$, and 4 and the points $(-1, -2)$, $(1, 3)$, and $(0, 5)$.

Planar lamina - thin, flat plate of material of constant density

Usually, Density = $\frac{\text{Mass}}{\text{Volume}}$, but for a planar lamina, Density = $\frac{\text{Mass}}{\text{Area}} = \rho$ (rho)

Consider an irregularly shaped planar lamina, bounded by $f(x)$ and $g(x)$ between a and b:

Mass = (density)(area)

To find the center of mass of the lamina, partition the interval $[a, b]$ into n equal width subintervals.

Mass of ith rectangle = (density)(area)

Directed distance from x-axis to (x_i, y_i) is y_i =
(midpoint of y-values)

Moment = (mass)(distance)
about x-axis

Moment = (mass)(distance)
about y-axis

Moments and Center of Mass of a Planar Lamina:

Let $f(x) \geq g(x)$ on $[a, b]$, and let the planar lamina have uniform density ρ,

1. The **moments about the x- and y-axes** are

   $M_x = \rho\int_a^b [\frac{f(x)+g(x)}{2}][f(x) - g(x)]dx$

   $M_y = \rho\int_a^b x[f(x) - g(x)]dx$

2. The **center of mass ($\overline{x}$, $\overline{y}$)** is given by

   $\overline{x} = \frac{My}{m}$ and $\overline{y} = \frac{Mx}{m}$, where

   $m = \rho\int_a^b (f(x) - g(x))dx$ is the mass of the lamina.

Notice that the density, ρ, will be in both the numerator and denominator of the center of mass, so it will cancel out. So the center of mass of a lamina of *uniform* density depends only on the shape of the lamina and not on its density. For this reason, the point **($\overline{x}$, $\overline{y}$)** is called the center of mass of a region, or the **centroid** of the region. To find the centroid of a region, just assume that the region has a constant density of $\rho = 1$.

So, Area $A = \int_a^b (f(x) - g(x))dx$ and

$\overline{x} = \frac{1}{A}\int_a^b x(f(x) - g(x))dx$ $\qquad$ $\overline{y} = \frac{1}{A}\int_a^b (\frac{f(x)+g(x)}{2})(f(x) - g(x))dx$

or

$A = \int_c^d (f(y) - g(y))dy$ and

$\overline{x} = \frac{1}{A}\int_c^d (\frac{f(y)+g(y)}{2})(f(y) - g(y))dy$ $\qquad$ $\overline{y} = \frac{1}{A}\int_c^d y(f(y) - g(y))dy$

3. Find centroid of region bounded by y $= \sqrt{x}$, $y = 0$ and $x = 9$.

4. Find centroid of region bounded by $y = x^2$ and $x + y = 2$.

5. Find centroid of region bounded by $x = 9 - y^2$ and $x + y = 3$.

6. Introduce a coordinate system and find the centroid of the simple plane region shown below.

Theorem of Pappus: Let R be a region in a plane and let L be a line in the same plane such that L does not intersect the interior of R. If r is the distance between the centroid of R and the line, then the volume, V, of the solid of revolution form by revolving R about the line is $V = 2\pi r A$ where A is the area of R.

7. Use the Theorem of Pappus to find the volume of a 2 foot $\times$ 4 foot rectangle revolved about a line 3 feet to the right of the longer boundary.