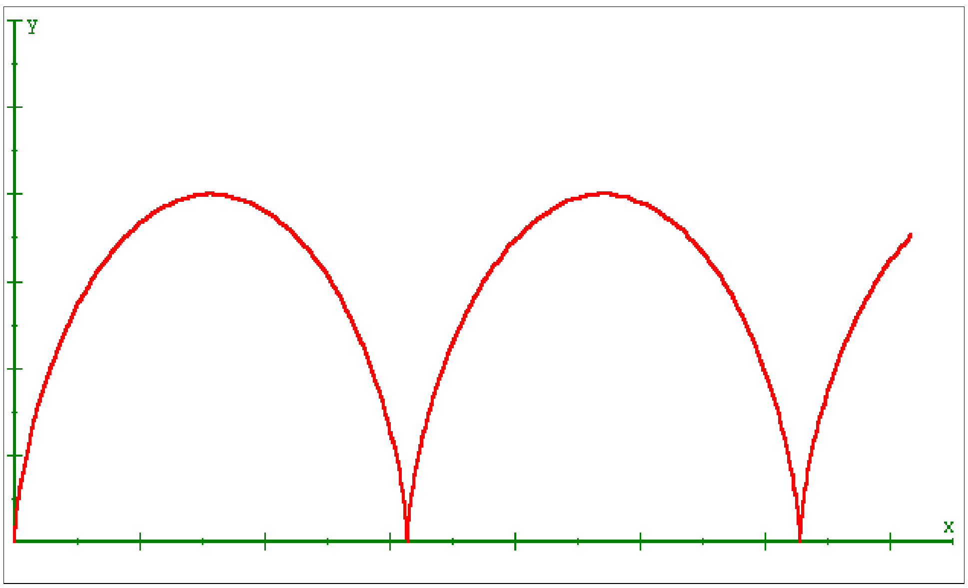
Parametric Equations and Calculus

A curve C represented by $x = f(t)$, $y = g(t)$ on an interval I is called **smooth** if $f'(t)$ and $g'(t)$ are continuous and not simultaneously 0, except possibly at the endpoints of I. The curve C is **piecewise smooth** if it is smooth on each subinterval of some partition of I.

Consider the parametric curve $x = t - \sin t, y = 1 - \cos t$. Find places where C is not smooth.

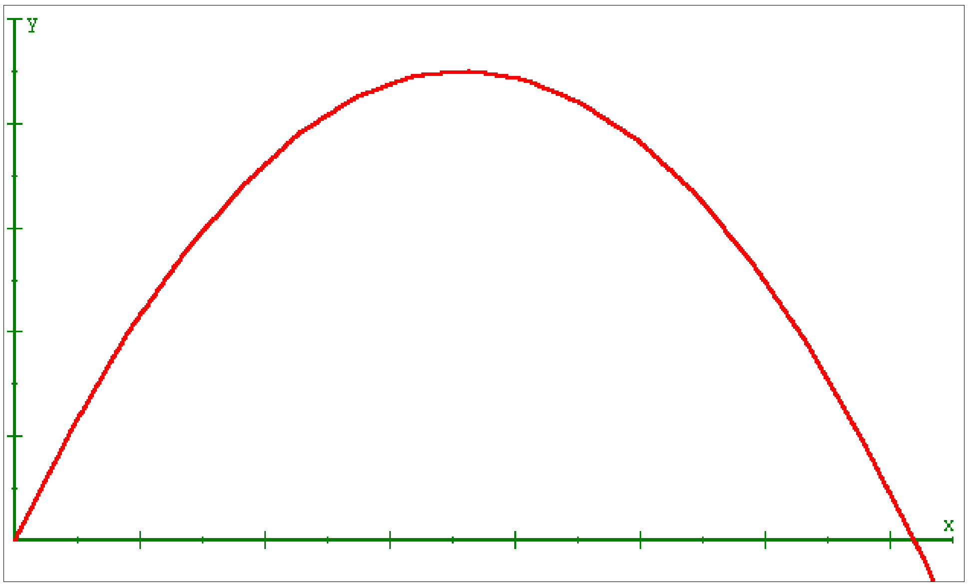


Use the equations of motion to represent parametrically the path of a projectile fired from the ground at an angle of $45°$ with an initial velocity of 48 feet per second. Recall the equations of motion:

$$x = (v_0 \cos\theta)t \qquad y = h + (v_0 \sin\theta)t - 16t^2.$$

$$\therefore \quad x = 48\,\frac{\sqrt{2}}{2}t \qquad y = 0 + 24\sqrt{2}t - 16t^2$$

We know that the object was projected at $45°$, but how can we find the angle at some other time or how can we find the maximum height?



The following theorem indicates how we can do that.

Theorem: If a smooth curve C is given by the equations $x = f(t)$ and $y = g(t)$, then the slope of C at (x, y) is $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}, \quad \dfrac{dx}{dt} \neq 0$

1. Find the derivative of the parametric equations: $x = t^3 + 1, y = t - 1$

2. Find the derivative of $x = 48\,\frac{\sqrt{2}}{2}t, y = 0 + 24\sqrt{2}t - 16t^2$ and use it to find the angle of elevation when

$t = 1$

Because $\dfrac{dy}{dx}$ is a function of $t,$ we can use the previous theorem repeatedly to find higher order derivatives. For example

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left[\frac{dy}{dx}\right] = \frac{\frac{d}{dt}\left[\frac{dy}{dx}\right]}{\frac{dx}{dt}} \qquad \frac{d^3y}{dx^3} = \frac{d}{dx}\left[\frac{d^2y}{dx^2}\right] = \frac{\frac{d}{dt}\left[\frac{d^2y}{dx^2}\right]}{\frac{dx}{dt}}$$

3. Find the first, second and third derivative of the parametric equations $x = t^2$, $y = 5t^3 - 3.$

4. Find the equation of the tangent line when $t = 1.$

Find vertical and horizontal tangent lines.

Since $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$, we can find horizontal tangent lines when $\frac{dy}{dx} = 0$ i.e. when $\frac{dy}{dt} = 0$ (Assuming $\frac{dx}{dt} \neq 0$).

Likewise we can find vertical tangent lines when $\frac{dy}{dx}$ is undefined, i.e. when $\frac{dx}{dt} = 0$ (Assuming $\frac{dy}{dt} \neq 0$)

If both $\frac{dy}{dt}$ and $\frac{dx}{dt} = 0$ at the same time, no conclusion can be drawn without investing further.

5. Find all the points of vertical and horizontal tangency to the curve $x = \cos t, y = 2\sin 2t$

Theorem: If a smooth curve C is given by $x = f(t)$ and $y = g(t)$ such that C does not intersect itself on the interval $a \leq t \leq b$ (except possibly at the endpoints), then the **length of the arc** of C over the interval is given by $\displaystyle\int_a^b \sqrt{(f'(t))^2 + (g'(t))^2}\, dt$ Note: When applying the arc length formula, be sure that the curve is traced out only once on the interval of integration.

Proof:

6. Find the length of the parametric curve described by $x = t^2, y = 4t^3 - 1$ on $[1, 2]$.

7. Find the length of the parametric curve described by $x = t^2, y = 4t^3 - 1$ on $[-1, 1]$.

Theorem: The **area** of a region bounded by a parametric curve $x = f(t)$, $y = g(t)$, $g(t) \geq 0,$ in which y is a continuous function of x on the interval $a \leq x \leq b$ where $a = f(t_1)$ and $b = f(t_2)$ and both $g(t)$ and $f'(t)$ are continuous on $[t_1, t_2]$ is given by $\displaystyle\int_a^b y\,dx = \int_{t_1}^{t_2} g(t)|f'(t)|\,dt.$

8. Find the area of one arch of $x = t - \sin t, y = 1 - \cos t$

Extra:

9. Find the area bounded by $x = \sqrt{4-t}, y = \sqrt{t}, x = 0$ and the x-axis