

Power Series and Interval of Convergence

We have learned that functions can be *approximated* by a Taylor or Maclaurin polynomial. We saw that the higher the degree of the polynomial the better the approximation became. In this section we learn that several of these functions can be represented *exactly* by an infinite series called a power series.

Definition: If x is a variable, then an infinite series of the form

$$\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + \dots + a_n x^n + \dots$$
 is called a power series.

More generally, a series of the form

$$\sum_{n=0}^{\infty} a_n (x - c)^n = a_0 + a_1 (x - c) + a_2 (x - c)^2 + a_3 (x - c)^3 + \dots + a_n (x - c)^n + \dots$$
 is called a power series centered at c .

A power series can be viewed as a function $f(x) = \sum_{n=0}^{\infty} a_n (x - c)^n$ having as its domain the values of x for which the power series converges.

Convergence of a power series

For a power series centered at c , precisely one of the following is true.

1. The series converges only at c .
2. There exists a real number $R > 0$ such that the series converges absolutely for $|x - c| < R$ and diverges for $|x - c| > R$.
3. The series converges absolutely for all x .

The number R is called the **radius of convergence**.

The set of all values of x for which the power series converges is called the **interval of convergence**.

Always use the Ratio Test to find the radius of convergence. Test the endpoints independently to obtain interval of convergence.

Ratio Test: Series converges if: $\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$

Find the radius of convergence on examples 1 - 3:

1.
$$\sum_{n=0}^{\infty} (4x)^n$$

2.
$$\sum_{n=0}^{\infty} \frac{(2x)^n}{n!}$$

3.
$$\sum_{n=0}^{\infty} n!x^n$$

Find the interval of convergence on examples 4 - 7:

4.
$$\sum_{n=0}^{\infty} (-1)^{n+1} n x^n$$

5.
$$\sum_{n=0}^{\infty} \frac{(-1)^n x^n}{(n+1)(n+2)}$$

6.
$$\sum_{n=0}^{\infty} \frac{(-1)^n n! (x - 4)^n}{3^n}$$

7.
$$\sum_{n=0}^{\infty} \frac{(x - 2)^{n+1}}{(n + 1)3^{n+1}}$$

Because power series are functions, they possess characteristics similar to other functions. We can differentiate and integrate power series as a way of obtaining a 'new' power series from 'old' power series.

Theorem: Properties of a Function Defined by a Power Series

If the function given by

$$f(x) = \sum_{n=0}^{\infty} a_n(x - c)^n = a_0 + a_1(x - c) + a_2(x - c)^2 + a_3(x - c)^3 + \dots$$

has a radius of convergence of $R > 0$, then on the interval $(c - R, c + R)$, $f(x)$ is continuous, differentiable and integrable where

$$f'(x) = \sum_{n=1}^{\infty} n a_n(x - c)^{n-1} = a_1 + 2a_2(x - c) + 3a_3(x - c)^2 + 4a_4(x - c)^3 + \dots$$

and

$$\int f(x) dx = C + \sum_{n=0}^{\infty} a_n \frac{(x - c)^{n+1}}{n + 1} = C + a_0(x - c) + a_1 \frac{(x - c)^2}{2} + a_2 \frac{(x - c)^3}{3} + \dots$$

The radius of convergence for the series obtained by differentiating or integrating a power series is the same as for the original power series. The interval of convergence may differ at the endpoints. When differentiating you will not gain endpoints; when integrating you will not lose endpoints.

8. Given the power series $f(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-5)^n}{n 5^n}$, find the interval of convergence.

Find a power series for $f'(x)$ and the interval of convergence for $f'(x)$

$$f(x) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-5)^n}{n 5^n}$$

Find a power series for $f''(x)$ and the interval of convergence for $f''(x)$

Find a power series for $\int f(x) dx$ and the interval of convergence.

10. Show that the function $y = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$ is a solution of the differential equation $y'' + y = 0$.