## Ratio and Root Tests

We discuss two tests in this section that are particularly useful for infinite series which converge rapidly. Series involving factorials, powers, roots, and infinite products are of this type. The tests typically do NOT work on infinite series whose general terms are rational expressions such as $\sum_{n=1}^{\infty} \frac{1}{n}$ or $\sum_{n=1}^{\infty} \frac{3n+2}{5n^3-1}$. Both of these tests allow us to conclude in certain situations that the series are Absolutely Convergent. This may be a stronger result than desired. If just checking for convergence we will conclude absolute convergence by these tests and hence convergence by the Absolute Convergence Theorem. Neither of these tests provide any information about the sum of the convergent series. The statements of the tests follow.

Ratio Test:

Let $\sum_{n=1}^{\infty} a_n$ be a series with non-zero terms. There are three possible conclusions from the Ratio Test.

**1.** $\sum_{n=1}^{\infty} a_n$ converges absolutely if $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$

**2.** $\sum_{n=1}^{\infty} a_n$ diverges if $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| > 1$ or $\infty$

**3.** No conclusion about $\sum_{n=1}^{\infty} a_n$ if $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$

Root Test:

$\sum_{n=1}^{\infty} a_n$ be a series with non-zero terms. There are three possible conclusions from the Root Test.

**1.** $\sum_{n=1}^{\infty} a_n$ converges absolutely if $\lim_{n \to \infty} \sqrt[n]{|a_n|} < 1$

**2.** $\sum_{n=1}^{\infty} a_n$ diverges if $\lim_{n \to \infty} \sqrt[n]{|a_n|} > 1$ or $\infty$

**3.** No conclusion about $\sum_{n=1}^{\infty} a_n$ if $\lim_{n \to \infty} \sqrt[n]{|a_n|} = 1$

Examples:

**1.** Use the Ratio Test on $\sum_{n=1}^{\infty} \frac{1}{n^2}$ and on $\sum_{n=1}^{\infty} \frac{1}{n}$ to show that the test is inconclusive when $\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = 1$

Determine if each of the following infinite series converges or diverges. Provide detailed and clear rationale for your decision.

**2.** $$\sum_{n=1}^{\infty} n\left(\frac{2}{3}\right)^n$$

**3.** $$\sum_{n=1}^{\infty} \frac{n^2}{2^n}$$

**4.** $$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n+2}{n(n+1)}$$

**5.** $$\sum_{n=1}^{\infty} \frac{2n!}{n^5}$$

**6.** $$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{2^{4n}}{(2n+1)!}$$

**7.** $$\sum_{n=1}^{\infty}(-1)^n \frac{2 \bullet 4 \bullet 6 \bullet \ldots \bullet (2n)}{2 \bullet 5 \bullet 8 \bullet \ldots \bullet (3n-1)}$$

**8.** $\displaystyle\sum_{n=1}^{\infty} \frac{e^{2n}}{n^n}$

**9.** $\displaystyle\sum_{n=1}^{\infty} e^{-n}$

Carefully study your textbook for a summary and a strategy for testing infinite series for convergence or divergence. There should be a table summary of all the tests for divergence and convergence.