

Taylor Polynomials and Approximations

Our objective in this section is to find a polynomial approximation for a non-polynomial function. We will start with an example.

1. We want to find a polynomial function, $P_n(x)$ which best approximates $f(x) = \cos x$ near 0.

What features should $P_n(x)$ have?

Linear approximation:

$$f(x) = \cos x$$

$$f(0) =$$

$$f'(x) =$$

$$f'(0) =$$

$$P_1(x) = mx + b$$

$$P_1(0) =$$

$$P_1'(x) =$$

$$P_1'(0) =$$

$$\therefore P_1(x) =$$

Quadratic approximation:

$$f(x) = \cos x$$

$$f(0) =$$

$$f'(x) =$$

$$f'(0) =$$

$$f''(x) =$$

$$f''(0) =$$

$$P_2(x) = ax^2 + bx + c$$

$$P_2(0) =$$

$$P_2'(x) =$$

$$P_2'(0) =$$

$$P_2''(x) =$$

$$P_2''(0) =$$

$$\therefore P_2(x) =$$

Fourth degree polynomial approximation:

$$f(x) = \cos x$$

$$P_4(x) = ax^4 + bx^3 + cx^2 + dx + e$$

$$f(0) =$$

$$P_4(0) =$$

$$f'(x) =$$

$$P_4'(x) =$$

$$f'(0) =$$

$$P_4'(0) =$$

$$f''(x) =$$

$$P_4''(x) =$$

$$f''(0) =$$

$$P_4''(0) =$$

$$f'''(x) =$$

$$P_4'''(x) =$$

$$f'''(0) =$$

$$P_4'''(0) =$$

$$f^{(4)}(x) =$$

$$P_4^{(4)}(x) =$$

$$f^{(4)}(0) =$$

$$P_4^{(4)}(0) =$$

$$\therefore P_4(x) =$$

	$\cos x$	$P_2(x)$	$P_4(x)$
.1			
.5			
.75			
.9			
1			

The higher the degree of the polynomial, the better the approximation.
The further you get from zero, the worse the approximation.

In general, to find a polynomial approximation $P_n(x)$ for $f(x)$ for values of x near zero, we will construct $P_n(x)$ so that $P_n^{(k)}(0) = f^{(k)}(0) \forall k$ (orders of the derivative).

$$P_n(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + \dots + c_nx^n = \sum_{k=0}^n c_k x^k$$

We need to find the coefficients (c_i)

$$P_n(x) =$$

$$P'_n(x) =$$

$$P''_n(x) =$$

$$P'''_n(x) =$$

$$P_n^{(4)}(x) =$$

$$P_n^{(5)}(x) =$$

$$P_n^{(n)}(x) =$$

$$P_n^{(n)}(0) =$$

$$\therefore C_n =$$

$$\therefore P_n(x) =$$

This is called the **Maclaurin polynomial** of $f(x)$ at 0. It works well to approximate values of $f(x)$ near 0.

We can show that $P_n(x) = \sum_{k=0}^n \frac{f^{(k)}(a)}{k!} (x - a)^k$ works well to approximate values near a . This is called the **Taylor polynomial** of $f(x)$ centered at a .
 (The Maclaurin polynomial uses $a = 0$.)

2. Find the 4th degree Maclaurin polynomial for $f(x) = \cos x$.

Recall that $P_n(x) = \sum_{k=0}^n \frac{f^{(k)}(0)}{k!} x^k$, Therefore $P_4(x) = \sum_{k=0}^4 \frac{f^{(k)}(0)}{k!} x^k$

$$P_4(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4$$

$$f(x) = \cos x$$

$$f(0) =$$

$$f'(x) =$$

$$f'(0) =$$

$$f''(x) =$$

$$f''(0) =$$

$$f'''(x) =$$

$$f'''(0) =$$

$$f^{(4)}(x) =$$

$$f^{(4)}(0) =$$

Therefore $P_4(x) =$

3. Find the n th degree Maclaurin polynomial for $f(x) = e^x$

Recall that
$$P_n(x) = \sum_{k=0}^n \frac{f^{(k)}(0)}{k!} x^k,$$

$$P_n(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)}{n!}x^n$$

$$f(x) = e^x$$

$$f(0) =$$

$$f'(x) =$$

$$f'(0) =$$

$$f''(x) =$$

$$f''(0) =$$

$$f'''(x) =$$

$$f'''(0) =$$

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$$f^{(k)}(x) =$$

$$f^{(k)}(0) =$$

Therefore $P_n(x) =$

4. Find the 4th degree Taylor polynomial for $f(x) = \ln x$ at 1.

Recall that
$$P_4(x) = \sum_{k=0}^4 \frac{f^{(k)}(1)}{k!} (x - 1)^k,$$

$$P_4(x) = f(1) + \frac{f'(1)}{1!}(x - 1) + \frac{f''(1)}{2!}(x - 1)^2 + \frac{f'''(1)}{3!}(x - 1)^3 + \frac{f^{(4)}(1)}{4!}(x - 1)^4$$

$$f(x) = \ln x$$

$$f(1) =$$

$$f'(x) =$$

$$f'(1) =$$

$$f''(x) =$$

$$f''(1) =$$

$$f'''(x) =$$

$$f'''(1) =$$

$$f^{(4)}(x) =$$

$$f^{(4)}(1) =$$

$$\therefore P_4(x) =$$

$$\ln 1.1 =$$

$$P_4(1.1) =$$

$$\ln 1.5 =$$

$$P_4(1.5) =$$

5. Find the 3th degree Maclaurin polynomial for $f(x) = \sin \pi x$.

Recall that $P_n(x) = \sum_{k=0}^n \frac{f^{(k)}(0)}{k!} x^k$, Therefore $P_3(x) = \sum_{k=0}^3 \frac{f^{(k)}(0)}{k!} x^k$

$$P_3(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3$$

$$f(x) = \sin \pi x$$

$$f(0) =$$

$$f'(x) =$$

$$f'(0) =$$

$$f''(x) =$$

$$f''(0) =$$

$$f'''(x) =$$

$$f'''(0) =$$

Therefore $P_3(x) =$

6. Find the 4th degree Taylor polynomial for $f(x) = \sqrt{x}$ at 4.

Recall that
$$P_4(x) = \sum_{k=0}^4 \frac{f^{(k)}(4)}{k!} (x - 4)^k,$$

$$P_4(x) = f(4) + \frac{f'(4)}{1!}(x - 4) + \frac{f''(4)}{2!}(x - 4)^2 + \frac{f'''(4)}{3!}(x - 4)^3 + \frac{f^{(4)}(4)}{4!}(x - 4)^4$$

$$f(x) = \sqrt{x}$$

$$f(4) =$$

$$f'(x) =$$

$$f'(4) =$$

$$f''(x) =$$

$$f''(4) =$$

$$f'''(x) =$$

$$f'''(4) =$$

$$f^{(4)}(x) =$$

$$f^{(4)}(4) =$$

$$\therefore P_4(x) =$$

An approximation is of little or no value unless we have some idea as to its accuracy. To measure the accuracy of a Taylor or Maclaurin polynomial we consider the concept of its remainder.

$f(x)$	=	$P_n(x)$	+	$R_n(x)$
Exact Value		Approximate Value		Remainder using $P_n(x)$

$$\begin{aligned} \therefore R_n(x) &= f(x) - P_n(x) \\ |R_n(x)| &= |f(x) - P_n(x)| \end{aligned}$$

Taylor's Theorem with the Lagrange form of the remainder

If a function $f(x)$ is differentiable through order $n + 1$ in an interval I containing c , then for each x in I , \exists a z between x and c such that

$$f(x) = f(c) + f'(c)(x - c) + \frac{f''(c)}{2!}(x - c)^2 + \dots + \frac{f^{(n)}(c)}{n!}(x - c)^n + R_n(x)$$

where $R_n(x) = \frac{f^{(n+1)}(z)}{(n + 1)!}(x - c)^{n+1}$

(Usually we are not able to find z , but we can usually find bounds for $f^{(n+1)}(z)$ to find the size of the remainder. The remainder is typically much smaller than the bounds we find.)

7. Use Taylor's Theorem to approximate e^1 by using a 5th degree Maclaurin polynomial and determine the accuracy of the approximation.

$$f(x) = e^x$$

$$f^{(k)}(x) = e^x$$

$$f^{(k)}(0) = 1$$

Recall that
$$P_n(x) = \sum_{k=0}^n \frac{f^{(k)}(0)}{k!} x^k$$

$$\therefore e^x \approx P_5(x) =$$

$$\therefore e^1 \approx P_5(1) =$$

$$R_5(x) =$$

8. Use a 3rd degree Maclaurin polynomial to approximate $\tan^{-1}(.5)$ and determine the accuracy of the approximation.

$$f(x) = \tan^{-1}(x) \qquad f(0) =$$

$$f'(x) = \frac{1}{1+x^2} \qquad f'(0) =$$

$$f''(x) = \frac{-2x}{(x^2+1)^2} \qquad f''(0) =$$

$$f'''(x) = \frac{2(3x^2-1)}{(x^2+1)^3} \qquad f'''(0) =$$

$$\therefore P_3(x) = \qquad \qquad \qquad \therefore P_3(.5) =$$

$$R_3(x) = \qquad \qquad \qquad \therefore R_3(.5) =$$

We need to bound $f^{(4)}(z)$

$$\therefore R_3(x) \leq$$

9. Determine the degree of the Maclaurin polynomial so that the error in approximating $e^{.75}$ is less than .0001.

$$f(x) = e^x \qquad f^{(n+1)}(x) = e^x$$

$$\begin{aligned} R_n(x) &= \frac{f^{(n+1)}(z)}{(n+1)!} x^{n+1} \\ &= \frac{e^z}{(n+1)!} x^{n+1} \quad \text{Since } 0 < z < .75 \Rightarrow e^0 < e^z < e^{.75} < e^1 < 3 \\ &< \frac{3}{(n+1)!} x^{n+1} \end{aligned}$$

$$\therefore R_n(.75) < \frac{3(.75)^{n+1}}{(n+1)!} \text{ and we need this to be less than .0001}$$

- 10.** Determine the value of x such that $f(x) = \sin x$ can be replaced by its 3rd degree Maclaurin polynomial if the error is be less than .0001.