Taylor Polynomials and Approximations

Our objective in this section is to find a polynomial approximation for a non-polynomial function. We will start with an example.

1. We want to find a polynomial function, $P_n(x)$ which best approximates $f(x) = \cos x$ near 0.

What features should $P_n(x)$ have?

Linear approximation:

$f(x) = \cos x$	$P_1(x) = mx + b$
$f(0) =$	$P_1(0) =$
$f'(x) =$	$P_1'(x) =$
$f'(0) =$	$P_1'(0) =$

$\therefore \quad P_1(x) =$

Quadratic approximation:

$f(x) = \cos x$	$P_2(x) = ax^2 + bx + c$
$f(0) =$	$P_2(0) =$
$f'(x) =$	$P_2'(x) =$
$f'(0) =$	$P_2'(0) =$
$f''(x) =$	$P_2''(x) =$
$f''(0) =$	$P_2''(0) =$

$\therefore \quad P_2(x) =$

Fourth degree polynomial approximation:

$f(x) = \cos x$ $\qquad$ $P_4(x) = ax^4 + bx^3 + cx^2 + dx + e$

$f(0) =$ $\qquad$ $P_4(0) =$

$f'(x) =$ $\qquad$ $P_4'(x) =$

$f'(0) =$ $\qquad$ $P_4'(0) =$

$f''(x) =$ $\qquad$ $P_4''(x) =$

$f''(0) =$ $\qquad$ $P_4''(0) =$

$f'''(x) =$ $\qquad$ $P_4'''(x) =$

$f'''(0) =$ $\qquad$ $P_4'''(0) =$

$f^{(4)}(x) =$ $\qquad$ $P_4^{(4)}(x) =$

$f^{(4)}(0) =$ $\qquad$ $P_4^{(4)}(0) =$

$\therefore \quad P_4(x) =$

	$\cos x$	$P_2(x)$	$P_4(x)$
.1			
.5			
.75			
.9			
1			

The higher the degree of the polynomial, the better the approximation.
The further you get from zero, the worse the approximation.

In general, to find a polynomial approximation $P_n(x)$ for $f(x)$ for values of x near zero, we will construct $P_n(x)$ so that $P_n^{(k)}(0) = f^{(k)}(0) \; \forall \; k$ (orders of the derivative).

$$P_n(x) = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + \; \ldots + c_nx^n \; = \sum_{k=0}^{n} c_kx^k$$

We need to find the coefficients (c_i)

$P_n(x) =$

$P_n'(x) =$

$P_n''(x) =$

$P_n'''(x) =$

$P_n^{(4)}(x) =$

$P_n^{(5)}(x) =$

$P_n^{(n)}(x) =$

$P_n^{(n)}(0) =$

$\therefore C_n =$

$\therefore \; P_n(x) =$

This is called the **<u>Maclaurin polynomial</u>** of $f(x)$ at 0. It works well to approximate values of $f(x)$ near 0.

We can show that $P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$ works well to approximate values near a. This is called the **<u>Taylor polynomial</u>** of $f(x)$ centered at a.

(The Maclaurin polynomial uses $a = 0$.)

2. Find the 4th degree Maclaurin polynomial for $f(x) = \cos x$.

Recall that $P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k$, Therefore $P_4(x) = \sum_{k=0}^{4} \frac{f^{(k)}(0)}{k!} x^k$

$$P_4(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4$$

$f(x) = \cos x$ | $f(0) =$

$f'(x) =$ | $f'(0) =$

$f''(x) =$ | $f''(0) =$

$f'''(x) =$ | $f'''(0) =$

$f^{(4)}(x) =$ | $f^{(4)}(0) =$

Therefore $P_4(x) =$

3. Find the nth degree Maclaurin polynomial for $f(x) = e^x$

Recall that $P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}\, x^k$,

$$P_n(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots + \frac{f^{(n)}(0)}{n!}x^n$$

$f(x) = e^x$ $\qquad$ $f(0) =$

$f'(x) =$ $\qquad$ $f'(0) =$

$f''(x) =$ $\qquad$ $f''(0) =$

$f'''(x) =$ $\qquad$ $f'''(0) =$

.
.
.

$f^{(k)}(x) =$ $\qquad$ $f^{(k)}(0) =$

Therefore $P_n(x) =$

4. Find the 4th degree Taylor polynomial for $f(x) = \ln x$ at 1.

Recall that $P_4(x) = \sum_{k=0}^{4} \frac{f^{(k)}(0)}{k!}(x-1)^k$,

$$P_4(x) = f(1) + \frac{f'(1)}{1!}(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 + \frac{f^{(4)}(1)}{4!}(x-1)^4$$

$f(x) = \ln x$ $\qquad$ $f(1) =$

$f'(x) =$ $\qquad$ $f'(1) =$

$f''(x) =$ $\qquad$ $f''(1) =$

$f'''(x) =$ $\qquad$ $f'''(1) =$

$f^{(4)}(x) =$ $\qquad$ $f^{(4)}(1) =$

$\therefore\ P_4(x) =$

$\ln 1.1 =$ $\qquad$ $P_4(1.1) =$

$\ln 1.5 =$ $\qquad$ $P_4(1.5) =$

5. Find the 3th degree Maclaurin polynomial for $f(x) = \sin \pi x$.

Recall that $P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}\, x^k$, Therefore $P_3(x) = \sum_{k=0}^{3} \frac{f^{(k)}(0)}{k!}\, x^k$

$$P_3(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3$$

$f(x) = \sin \pi x$ $\qquad$ $f(0) =$

$f'(x) =$ $\qquad$ $f'(0) =$

$f''(x) =$ $\qquad$ $f''(0) =$

$f'''(x) =$ $\qquad$ $f'''(0) =$

Therefore $P_3(x) =$

6. Find the 4th degree Taylor polynomial for $f(x) = \sqrt{x}$ at 4.

Recall that $P_4(x) = \sum_{k=0}^{4} \frac{f^{(k)}(4)}{k!} (x-4)^k$,

$$P_4(x) = f(4) + \frac{f'(4)}{1!}(x-4) + \frac{f''(4)}{2!}(x-4)^2 + \frac{f'''(4)}{3!}(x-4)^3 + \frac{f^{(4)}(4)}{4!}(x-4)^4$$

$f(x) = \sqrt{x}$ $\qquad$ $f(4) =$

$f'(x) =$ $\qquad$ $f'(4) =$

$f''(x) =$ $\qquad$ $f''(4) =$

$f'''(x) =$ $\qquad$ $f'''(4) =$

$f^{(4)}(x) =$ $\qquad$ $f^{(4)}(4) =$

$\therefore\ P_4(x) =$

An approximation is of little or no value unless we have some idea as to its accuracy. To measure the accuracy of a Taylor or Maclaurin polynomial we consider the concept of its remainder.

$f(x)$	$=$	$P_n(x)$	$+$	$R_n(x)$
Exact Value		Approximate Value		Remainder using $P_n(x)$

$$\therefore \quad R_n(x) = f(x) - P_n(x)$$
$$|R_n(x)| = |f(x) - P_n(x)|$$

<u>Taylor's Theorem with the Lagrange form of the remainder</u>

If a function $f(x)$ is differentiable through order $n+1$ in an interval I containing c, then for each x in I, $\exists$ a z between x and c such that

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \ldots + \frac{f^{(n)}(c)}{n!}(x-c)^n + R_n(x)$$

where $R_n(x) = \dfrac{f^{(n+1)}(z)}{(n+1)!}(x-c)^{n+1}$

(Usually we are not able to find z, but we can usually find bounds for $f^{n+1}(z)$ to find the size of the remainder. The remainder is typically <u>much</u> smaller than the bounds we find.)

7. Use Taylor's Theorem to approximate e^1 by using a 5th degree Maclaurin polynomial and determine the accuracy of the approximation.

$f(x) = e^x \qquad f^{(k)}(x) = e^x \qquad f^{(k)}(0) = 1$

Recall that $P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!}\, x^k$

$\therefore \quad e^x \approx P_5(x) =$

$\therefore \quad e^1 \approx P_5(1) =$

$R_5(x) =$

8. Use a 3rd degree Maclaurin polynomial to approximate $\tan^{-1}(.5)$ and determine the accuracy of the approximation.

$f(x) = \tan^{-1}(x)$ $\qquad$ $f(0) =$

$f'(x) = \dfrac{1}{1+x^2}$ $\qquad$ $f'(0) =$

$f''(x) = \dfrac{-2x}{(x^2+1)^2}$ $\qquad$ $f''(0) =$

$f'''(x) = \dfrac{2(3x^2-1)}{(x^2+1)^3}$ $\qquad$ $f'''(0) =$

$\therefore\ P_3(x) =$ $\qquad$ $\therefore\ P_3(.5) =$

$R_3(x) =$ $\qquad$ $\therefore\ R_3(.5) =$

We need to bound $f^{(4)}(z)$

$\therefore\ R_3(x) \leq$

9. Determine the degree of the Maclaurin polynomial so that the error in approximating $e^{.75}$ is less than $.0001$.

$f(x) = e^x \qquad\qquad f^{(n+1)}(x) = e^x$

$$\begin{aligned} R_n(x) &= \frac{f^{(n+1)}(z)}{(n+1)!}x^{n+1} \\ &= \frac{e^z}{(n+1)!}x^{n+1} \qquad Since\, 0 < z < .75 \Rightarrow e^0 < e^z < e^{.75} < e^1 < 3 \\ &< \frac{3}{(n+1)!}x^{n+1} \end{aligned}$$

$\therefore\ R_n(.75) < \dfrac{3(.75)^{n+1}}{(n+1)!}$ and we need this to be less than $.0001$

10. Determine the value of x such that $f(x) = \sin x$ can be replaced by its 3rd degree Maclaurin polynomial if the error is be less than $.0001$.