Taylor and Maclaurin Series

We can use the same process we used to find a Taylor or Maclaurin polynomial to find a power series for a particular function as long as the function has infinitely many derivatives. The following theorem gives the form that every convergent power series must take.

If $f(x)$ is represented by a power series $\sum_{n=0}^{\infty} a_n (x-c)^n$ for all x in an open interval I containing c, then $a_n = \frac{f^{(n)}(c)}{n!}$ and

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2!}(x-c)^2 + \ \ldots + \frac{f^{(n)}(c)}{n!}(x-c)^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

This theorem is justified by repeating the process developed for a Taylor polynomial an infinite number of times.

The series obtained is called the Taylor series for $f(x)$ at c. If $c = 0$ then series is called the Maclaurin series for $f(x)$.

One can always find a Taylor series for a function if the function has an infinite number of derivatives at c. We can also find the interval of convergence for the series obtained. In most cases the series will converge to the function from which it was derived on its interval of convergence. However, sometimes the series will converge, but to a different function.

As we develop power series for non-algebraic functions, it will be important to show that our series actually converges to the intended function.

Two of the useful series obtained in the last section,
$\ln\left(\frac{1+x}{1-x}\right) = 2\sum_{n=0}^{\infty}\frac{x^{2n+1}}{2n+1} \quad |x| < 1$ and $\tan^{-1}x = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{2n+1} \quad |x| \le 1$,
were both derived using integration and/or substitution on the geometric series $\sum_{n=0}^{\infty} x^n$, which was already known to have sum $S = \frac{1}{1-x}$.

Therefore, both of these series not only converge, but converge to the function which they represent. Most of the series we developed in the last section by re-centering, substitution, integration or differentiation started with a geometric series which will always converge to the function it represents and any new series created from the geometric series will also converge to the intended function.

1. Derive a Maclaurin series for $f(x) = \sin x$ and determine the interval of convergence.

$f(x) = \sin x$	$f(0) =$
$f'(x) =$	$f'(0) =$
$f''(x) =$	$f''(0) =$
$f'''(x) =$	$f'''(0) =$
$f^{(4)}(x) =$	$f^{(4)}(0) =$
$f^{(5)}(x) =$	$f^{(5)}(0) =$

Maclaurin series for $\sin x$:

2. Find the Maclaurin series for $\cos x$ by differentiating the power series for $\sin x$.

3. Find the Maclaurin series for e^x.

All of these can be easily shown to be convergent for all real numbers, using the Ratio Test. However the power series may or may not converge to the function from which they were derived.

Again, it is not necessarily true that a Taylor (Maclaurin) series converges to the function from which it was derived. There could be functions which have the same derivatives at c, but are different elsewhere. Remember the coefficients of the power series are the values of the derivative of the function at c. $f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(c)}{n!}(x-c)^n$

4. Find the Maclaurin series for the function $g(x)=\begin{cases}-1 & x<-\frac{\pi}{2}\\ \sin x & -\frac{\pi}{2}\le x\le\frac{\pi}{2}\\ 1 & x>\frac{\pi}{2}\end{cases}$

This function will have the same Maclaurin series as does $f(x)=\sin x$. However, the obtained series cannot converge to both functions. We want to be able to determine what the series we derived converges to.

Theorem: Convergence of Taylor Series
If a function f has derivatives of all orders in an open interval I centered at c, then $f(x)=\sum_{n=0}^{\infty}\frac{f^{(n)}(c)}{n!}(x-c)^n$ (i.e. the Taylor series) converges to the function from which it was derived iff $\lim_{n\to\infty}R_n(x)=0$ for every x in I.

If a power series converges to $f(x)$, then the series must be a Taylor series even though it may have been obtained using some other method than Taylor's formula. However, every series formed using the Taylor series formula will not necessarily converge to $f(x)$.

5. Show that the series $\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$ converges to $\sin x$. That is, show that $\lim_{n\to\infty} R_n(x) = 0$.

6. Show that $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges to e^x.

Using the list of given power series as a start, create a power series for each of the following.

7. $f(x) = \sin x^2$

8. $g(x) = x \cos x$

9. $\boldsymbol{g(x)} = \cosh x$

10. $f(x) = \dfrac{e^x - 1}{x}$

11. Find the first 3 terms of a power series for $f(x) = e^x \cos x$.

12. Find a power series representation of $\dfrac{\sin x}{x}$ and use it to find $\lim_{x \to 0} \dfrac{\sin x}{x}$.

13. Verify that $\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{1}{n!}\right) = \frac{e-1}{e}$ and use the series to approximate the sum with an error less than .0001.

14. Use power series to approximate $\displaystyle\int_{.5}^{1} \cos\sqrt{x}\,dx$ with error $< .00001$

Consider Maclaurin series for $\sin x$, $\cos x$, e^x. These converge for all x, but converge slowly for values of x away from the center point of 0.

15. Find a Taylor series for $f(x) = e^x$ centered at 1.

16. Find a Taylor series for 3^x centered at 2.

The **binomial series** is a series used to represent a function in the form of $f(x) = (1+x)^k$. The binomial series is useful for computing roots.

17. Derive a Maclaurin series representation for the function $f(x) = (1+x)^k$.

$f(x) = (1+x)^k$ $\qquad$ $f(0)$

$f'(x) = k(1+x)^{k-1}$ $\qquad$ $f'(0)$

$f''(x) = k(k-1)(1+x)^{k-2}$ $\qquad$ $f''(0)$

$f'''(x) = k(k-1)(k-2)(1+x)^{k-3}$ $\qquad$ $f'''(0)$

$f^{(n)}(0)$

$\therefore$ the series representing $f(x) = (1+x)^k$ is

It can be shown using the Ratio Test that the radius of convergence is 1 and that $\lim_{n\to\infty} |R_n(x)| = 0$, hence this series converges to $f(x)$ for all $|x| < 1$.

18. Write a Maclaurin series for $f(x) = \sqrt{4 + x^2}$

19. Calculate $\sqrt{26}$ using a 3rd degree polynomial.

20. Find a series for $f(x) = \sin^{-1} x$ by integrating series for $\dfrac{1}{\sqrt{1-x^2}}$.