Conic Sections Video Lecture

Sections 10.1-10.4

Course Learning Objectives:
Graph conic sections and solve applications involving them.

Weekly Learning Objectives:

1) Know the names of the Conics.
2) Analyze parabolas with vertex at the origin.
3) Analyze ellipses with center at the origin.
4) Analyze ellipses with center at (h,k).
5) Analyze hyperbolas with center at the origin.
6) Find the asymptotes of a hyperbola.
7) Analyze hyperbolas with center at (h,k).
8) Solve applied problems involving conic sections.

Conic Sections

Conic sections are the curves formed by the intersection of a plane with the two nappes of a cone.

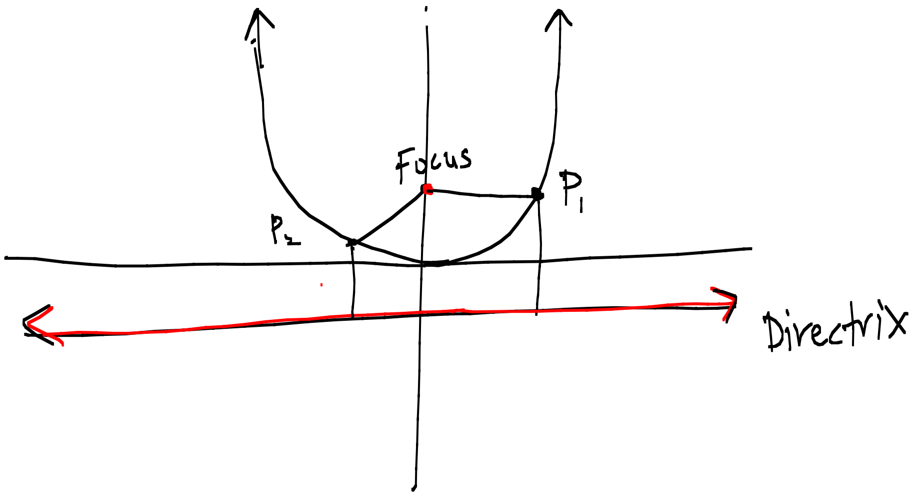
We have already studied parabolas from an algebraic point of view. We know that the equation $f(x) = ax^2 + bx + c$ has a set of solutions that lie on a $\cup$ shape. We can determine, depending on the values of $a, b,$ and c, whether the parabola opens up or down, where the vertex is, where the intercepts are, as well as finding additional points on the graph.

$a > 0$ $a < 0$

Vertex Formula: $\left(-\frac{b}{2a}, f\left(-\frac{b}{2a}\right)\right)$

We will now define the conic sections from a geometric point of view and show that these definitions are consistent with the algebraic discussion that we have had previously.

A <u>parabola</u> is the set of points equidistant from a given point (focus) and a given line (directrix).

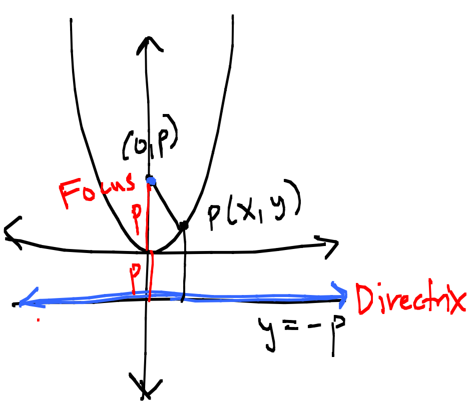


Derivation of the equation of a parabola with a vertical axis.

Start with a parabola centered at the origin. Let $P(x, y)$ be any point on the parabola, then the distance between P and the focus F is:

Distance from P to the directrix is:

By the definition of a parabola, these distances must be equal:



Parabola with vertical axis

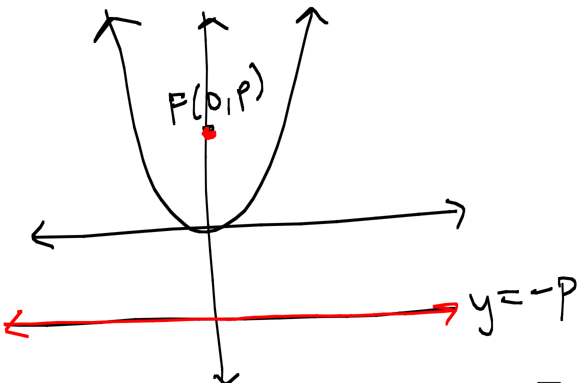
$$x^2 = 4py \quad \text{or} \quad y = \frac{1}{4p}x^2$$

Vertex $V(0,0)$

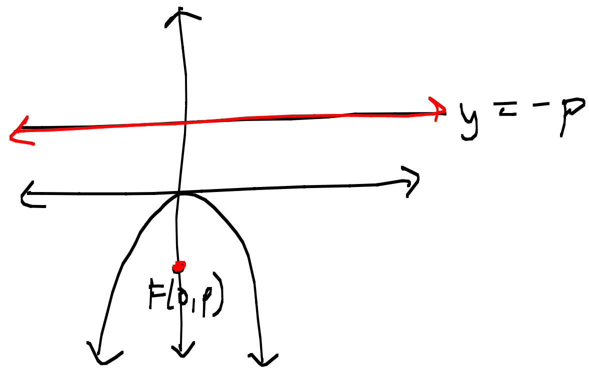
Focus $F(0,p)$

Directrix $y = -p$

If $p > 0$, opens up



If $p < 0$, opens down



Focal Diameter = $4p$

Parabola with horizontal axis

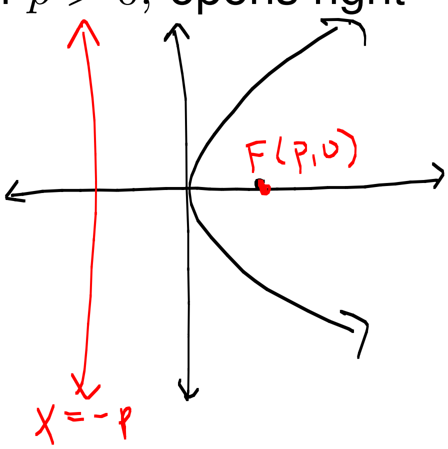
$$y^2 = 4px \quad \text{or} \quad x = \frac{1}{4p}y^2$$

Vertex $V(0,0)$

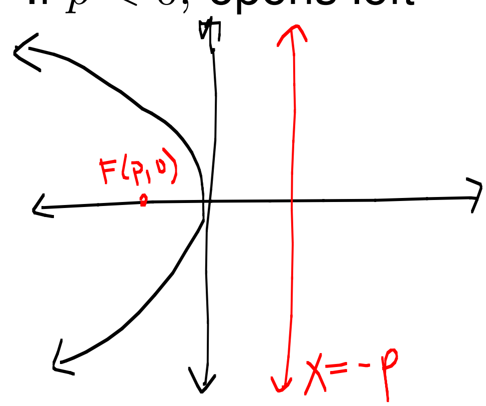
Focus $F(p,0)$

Directrix $x = -p$

If $p > 0$, opens right



If $p < 0$, opens left



Focal Diameter = $4p$

Application: If you put a light source at the focus, then light rays would reflect off of a parabola. Headlights work this way.

Examples:

1. Find focus, directrix, and focal diameter of the parabola and sketch the graph.

a) $y = -2x^2$

b) $16x - y^2 = 0$

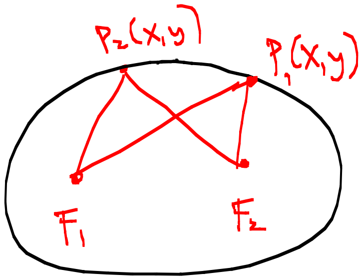
2. Find the equation of a parabola that satisfies the following conditions.

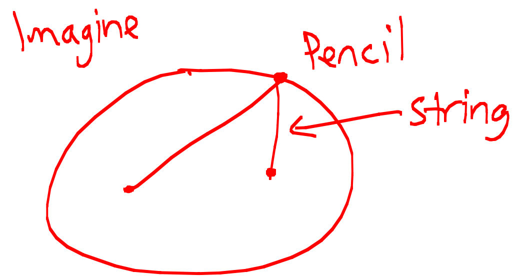
a) vertex at origin, Focus at $(5, 0)$

b) vertex at origin, Equation of Directrix: $y = 6$

c) Focal diameter 8, vertex at origin, Focus on the negative y-axis.

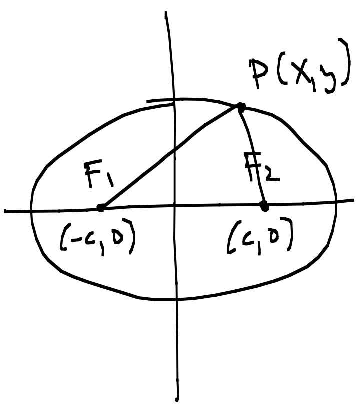
An <u>ellipse</u> is the set of all points in the plane the sum of whose distances form two fixed points F_1 and F_2 is a constant. These two points are called the foci (plural of focus) of the ellipse.





Derivation of formula of ellipse.

Let the sum of the distances from a point on the ellipse to the foci be $2a$, then:



Ellipse
centered at the origin

Equation:

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$a > b > 0$ (a is bigger than b)

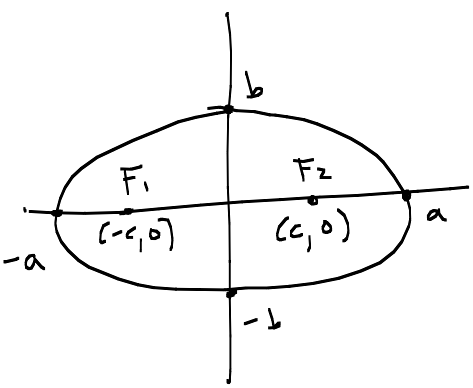
Vertices: $(\pm a, 0)$

Major Axis: Horizontal, length $2a$

Minor Axis: Vertical, length $2b$

Foci: $(\pm c, 0)$ $c^2 = a^2 - b^2$

Graph:



Equation:

$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$

$a > b > 0$

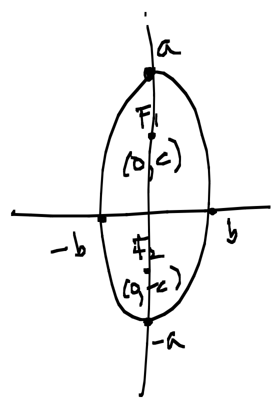
Vertices: $(0, \pm a)$

Major Axis: Vertical, length $2a$

Minor Axis: Horizontal, length $2b$

Foci: $(0, \pm c)$ $c^2 = a^2 - b^2$

Graph:



Examples:

3. Find the vertices, foci, length of major and minor axis, and sketch the graph for each of the following:

a) $\dfrac{x^2}{16} + \dfrac{y^2}{25} = 1$

b) $20x^2 + 4y^2 = 5$

c) $\frac{(x+2)^2}{4} + y^2 = 1$

d) $x^2 + 4y^2 + 20x - 40y + 300 = 0$

e) $x^2 + 4y^2 + 20x - 40y + 100 = 0$

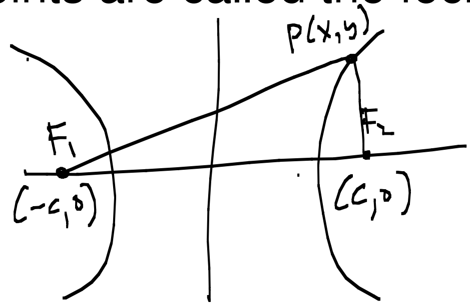
4. Write the equation of an ellipse satisfying the following conditions.

a) Foci at $(\pm 4, 0)$, vertices $(\pm 5, 0)$

b) Foci at $(0, \pm 5)$, length of major axis 12

c) Endpoints of the minor axis $(0, \pm 3)$, distance between foci is 6.

A <u>hyperbola</u> is the set of all points in the plane, the difference of whose distances from two fixed points F_1 and F_2 is a constant. The two fixed points are called the foci of the hyperbola.



Derivation of formula of hyperbola

Let the Foci be at $(\pm c, 0)$ and let the difference between the distances from the two foci is $\pm 2a$, then:

Hyperbola

Equation:

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ (x first)

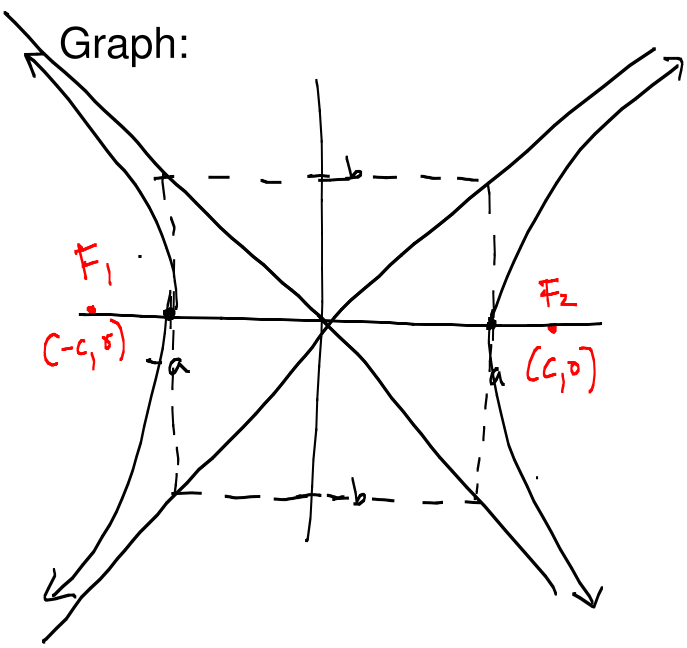
Vertices: $(\pm a, 0)$

Transverse Axis: Horizontal, length $2a$

Asymptotes: $y = \pm \frac{b}{a} x$

Foci: $(\pm c, 0)$ $c^2 = a^2 + b^2$

Graph:



Equation:

$\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ (y first)

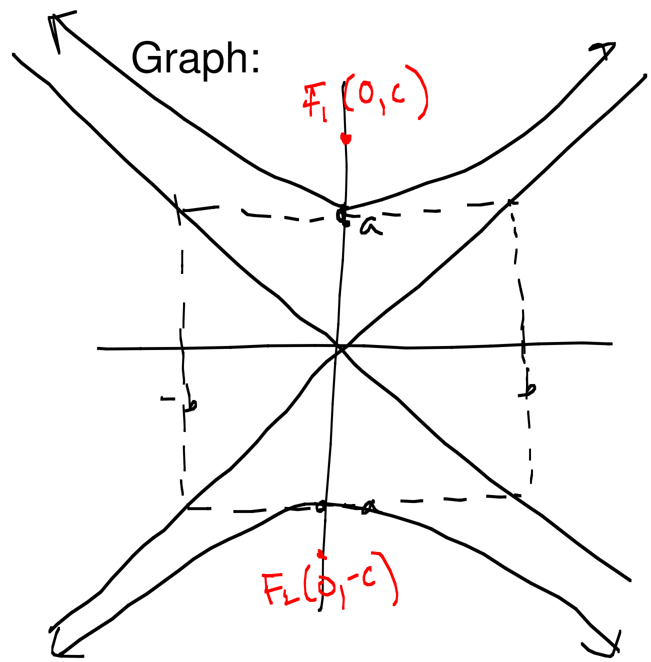
Vertices: $(0, \pm a)$

Transverse Axis: Vertical, length $2a$

Asymptotes: $y = \pm \frac{a}{b} x$

Foci: $(0, \pm c)$ $c^2 = a^2 + b^2$

Graph:



To graph:
1) Sketch the central box
2) Sketch the asymptotes
3) Plot vertices
4) Sketch 2 branches

Examples:

5. Find the vertices, foci, and asymptotes of the hyperbola and sketch its graph.

a) $\dfrac{y^2}{9} - \dfrac{x^2}{16} = 1$

b) $x^2 - 2y^2 = 3$

c) $\dfrac{(y-1)^2}{25} - (x+3)^2 = 1$

d) $x^2 - y^2 = 10x - 10y + 1$

6. Write the equation of a hyperbola satisfying the following conditions.

a) Foci $(0, \pm 10)$, vertices $(0, \pm 8)$

b) Vertices $(0, \pm 6)$, asymptotes $y = \pm \frac{1}{3}x$