Double and Half Angle Formulas Video Lecture

Sections 7.6 and 7.7

Course Learning Objectives:

1) Demonstrate an understanding of trigonometric functions and their applications.
2) Verify identities.

Weekly Learning Objectives:

1) Use double-angle formulas to find exact values.
2) Use half-angle formulas to find exact values.
3) Use double-angle and half-angle formulas to establish identities.
4) Express products as sums and use these formulas to establish identities.
5) Express sums as products and use these formulas to establish identities.
6) Reduce even powers of sine, cosine and tangent functions to a linear cosine function.

Double and Half Angle Identities

The identities developed in this section are a direct result of the sum and difference identities.

Recall that $\sin 2x \neq 2\sin x$. Think of one way that you can convince someone that $\sin 2x \neq 2\sin x$.

The relationships asked for above are stated in the double angle identities. The double angle identities are a direct result of the sum formulas.

How could we think of 2α in order to make use of the sum formula?

This is equivalent to $\mathbf{\sin(2\alpha) =}$

Use a similar strategy to develop an identity for cos (2α).

Use a similar strategy to develop an identity for tan (2α).

Double angle Formulas

$$\begin{aligned}
\sin(2\alpha) &= 2\sin\alpha\cos\alpha \\
\cos(2\alpha) &= \cos^2\alpha - \sin^2\alpha \\
&= 1 - 2\sin^2\alpha \\
&= 2\cos^2\alpha - 1 \\
\tan(2\alpha) &= \frac{2\tan\alpha}{1-\tan^2\alpha}
\end{aligned}$$

Note: These formulas are generalizations of specific formulas. The double angle formulas could be used to rewrite each of the following:

a) $\sin 10x$ b) $\cos 6x$ c) $\tan 100x$ d) $\sin \frac{x}{2}$

Examples:

1. If $\cos x = \dfrac{2}{7}, x \in Q\,IV$, find each of the following:

a) $\sin 2x$ b) $\cos 2x$ c) $\tan 2x$

2. If $\cot x = \dfrac{2}{3}$, where $\sin x < 0$, find each of the following.

a) $\sin 2x$ b) $\cos 2x$ c) $\tan 2x$

Solve $\cos 2\alpha = 2\cos^2\alpha - 1$ for $\cos^2\alpha$.

Solve $\cos 2\alpha = 1 - 2\sin^2\alpha$ for $\sin^2\alpha$.

Using the above 2 results, we can conclude that $\tan^2\alpha =$

These formulas allow us to reduce even powers of sine, cosine, and tangent functions to a linear function of the cosine function. This reduction is extremely important in calculus.

3. Write $f(x) = \cos^4 x \sin^2 x$ as a linear function of cosine.

Solving the equations on the previous page for $\sin\alpha$, $\cos\alpha$, and tan α yields, what are known as, the half-angle identities.

$$\sin^2 x = \frac{1-\cos 2x}{2} \qquad \cos^2 x = \frac{1+\cos 2x}{2} \qquad \tan^2 x = \frac{1-\cos 2x}{1+\cos 2x}$$

$\sin\alpha =$

$\cos\alpha =$

$\tan\alpha =$

Half-angle Formulas

$$\sin\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1-\cos\alpha}{2}}$$

$$\cos\left(\frac{\alpha}{2}\right) = \pm\sqrt{\frac{1+\cos\alpha}{2}}$$

$$\tan\left(\frac{\alpha}{2}\right) = \pm\frac{\sqrt{\frac{1-\cos\alpha}{2}}}{\sqrt{\frac{1+\cos\alpha}{2}}} = \frac{\sqrt{1-\cos\alpha}}{\sqrt{1+\cos\alpha}}$$

$$= \frac{1-\cos\alpha}{\sin\alpha}$$

$$= \frac{\sin\alpha}{1+\cos\alpha}$$

These are also generalizations. Use the half-angle formulas to rewrite each of the following:

a) $\sin 2x$ b) $\tan 6x$ c) $\cos \dfrac{x}{6}$

Examples:

4. Find the exact value for $\sin 15°$.

5. Find the exact value for $\cos \dfrac{\pi}{8}$.

6. Find the exact value for $2 \sin 75°\cos 75°$.

7. If $\cot x = 5$, where $180° < x < 270°$, find each of the following.

a) $\sin \dfrac{x}{2}$ b) $\cos \dfrac{x}{2}$ c) $\tan \dfrac{x}{2}$

Verify each of the following identities:

8. $\dfrac{2\tan x}{1+\tan^2 x} = \sin 2x$

There are several other identities, that while not worth memorizing, come in handy in rewriting trigonometric relationships. They are known as the product-to-sum formulas and the sum-to-product formulas. They are listed on the back flap of your text. You do not need to memorize these identities, but you do need to be able to use them to write products as sums or sums as products.

$\sin x + \sin y = 2\sin\frac{x+y}{2}\cos\frac{x-y}{2}$
$\sin x - \sin y = 2\sin\frac{x-y}{2}\cos\frac{x+y}{2}$
$\cos x + \cos y = 2\cos\frac{x+y}{2}\cos\frac{x-y}{2}$
$\cos x - \cos y = -2\sin\frac{x+y}{2}\sin\frac{x-y}{2}$

$\sin u \sin v = \frac{1}{2}[\cos(u-v) - \cos(u+v)]$
$\cos u \cos v = \frac{1}{2}[\cos(u-v) + \cos(u+v)]$
$\sin u \cos v = \frac{1}{2}[\sin(u+v) + \sin(u-v)]$

9. $$\frac{\sin 3x + \sin 7x}{\cos 3x - \cos 7x} = \cot 2x$$

10. Write as a sum: $\sin x \sin 5x$.

11. Write as a product: $\cos 5x + \cos 2x$

12. A rectangle is inscribed in a semicircle of radius 1. See the illustration.

a) Express the area of the rectangle as a function of the angle θ shown in the illustration.

b) Show that $A = \sin 2\theta$

c) Find the angle θ that results in the largest area A.

d) Find the dimensions of the largest rectangle.