Exponential Functions Video Lecture

Section 5.3

Course Learning Objectives:

1) Graph exponential functions and use such graphs to solve applied problems and to understand the significance of attributes of the graph to such applied problems.
2) Solve appropriate applications of determining compound interest and exponential growth and decay.
3) Identify and articulate the significance of graphical components in a mathematical model/application.

Weekly Learning Objectives:

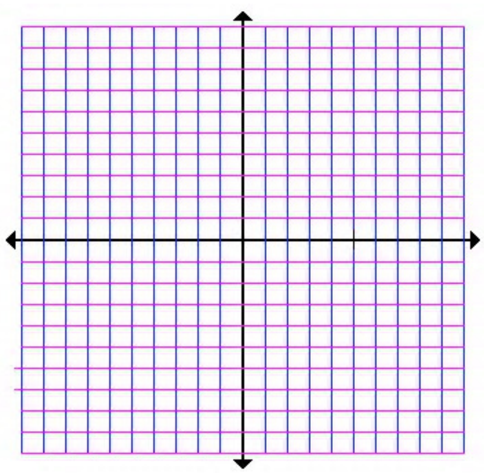
1) Evaluate exponential functions.
2) Graph exponential functions with transformations.
3) Define the number e.
4) Apply the compound interest formula to applications.
5) Determine the present value of a lump sum of money.
6) Apply the compound continuously formula to applications.
7) Solve exponential application problems including population and logistic growth.

Exponential Functions

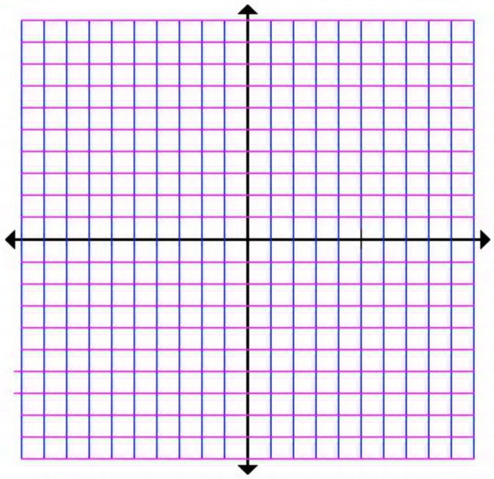
We will begin to study two of the most important functions in mathematics - the exponential function and its inverse, the logarithmic function. These functions can help us model a wide variety of natural phenomenon ranging from growth patterns in population or the value of money to radioactive decay, learning curves, spread of a virus or a rumor, carbon dating, and a host of other situations.

We will first look at the exponential function. The exponential function is a function of the form $f(x) = a^x,$ where $a > 0$ and x is a rational number.

1. Sketch the graph of $f(x) = 2^x$



2. Sketch the graph of $g(x) = \left(\frac{1}{2}\right)^x$



Characteristics of the graph of the exponential function $f(x) = a^x$

1. The graph of $f(x) = a^x$ will always contain the point $(0, 1)$.

2. The graph will always contain the points $(1, a)$ and $\left(-1, \frac{1}{a}\right)$

3. When $a > 1$ the graph is increasing. When $0 < a < 1$ the graph is decreasing.

4. The graph is asymptotic to the negative x-axis when $a > 1$. $(x \to -\infty, f(x) \to 0)$. The graph is asymptotic to the positive x-axis when $0 < a < 1$. $(x \to \infty, f(x) \to 0)$

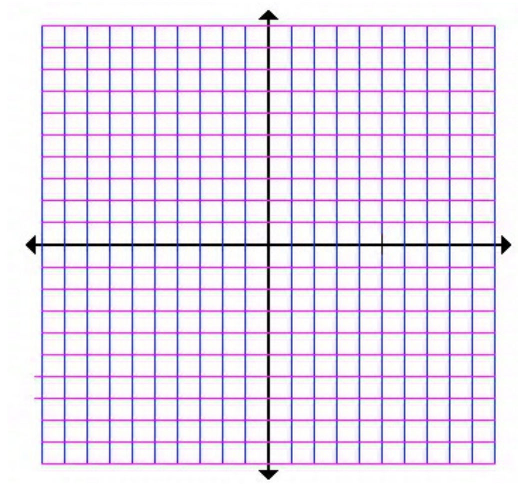
5. The domain is $(-\infty, \infty)$ and the range is $(0, \infty)$.

6. The function is one-to-one.

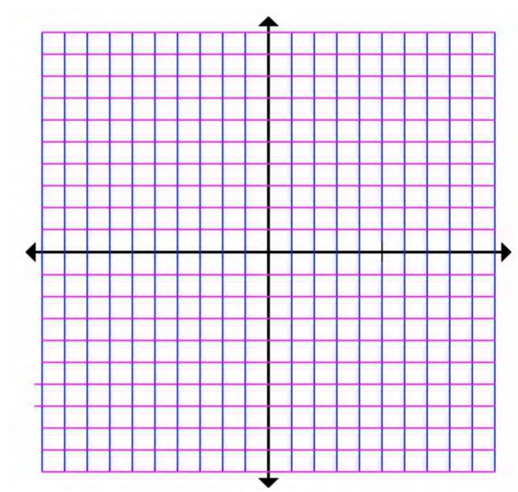
7. All previously discussed transformations apply to the graph of an exponential function.

Examples: Sketch the graph of each of the following indicating the transformations applied to the basic exponential function:

3. $f(x) = 3^{x-1} + 2$



4. $g(x) = -2^{2-x} - 3$



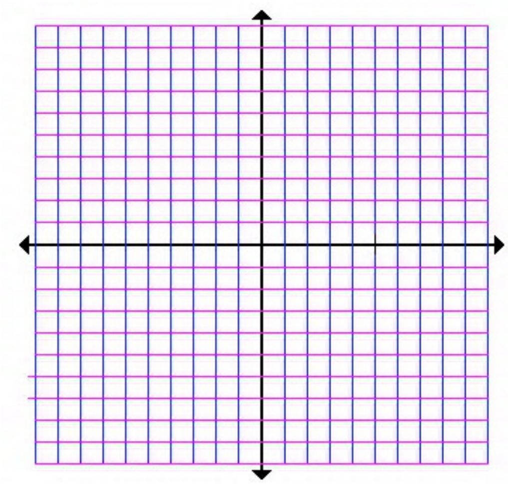
5. Find the equation of an exponential function of the form $f(x) = ca^x$ that passes through the points $(-1, 15)$ and $(0, 5)$.

We stated that any positive number can be used as the base in the exponential function. Bases 2 and 10 are convenient bases for many applications involving the exponential function but the most important base of the exponential function is the number e. The letter e is in recognition of a famous European mathematician of the 17^{th} century Leonard Euler. The letter e is defined to be the value of the expression $\left(1 + \frac{1}{n}\right)^n$ as $n \to \infty$. e is an irrational number.

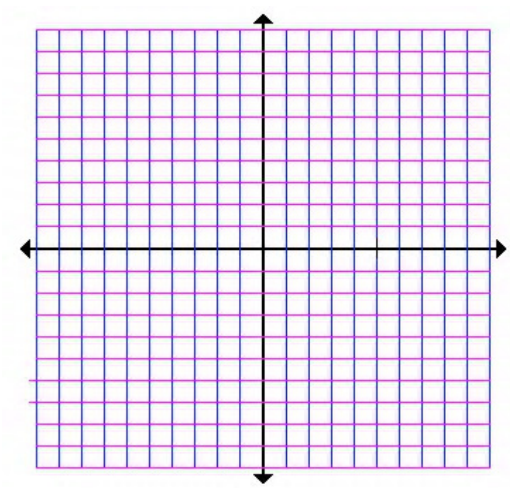
The natural exponential function is the exponential function $f(x) = e^x$. It is usually referred to as the exponential function. Note that the graph of $f(x) = e^x$ lies between the graph of $g(x) = 2^x$ and $h(x) = 3^x$.

Examples: Sketch the graph of each noting the transformations applied.

6. $f(x) = e^{-x} + 1$



7. $g(x) = e^{3-2x}$



Development of compound interest formula:

P = Prinicipal
i = interest rate per time period

After 1 time period, Interest =
Amount =

After 2 time periods, Interest =
Amount =

After k time periods, Amount =

If the <u>annual</u> interest rate is r compounded n times per year, then in each time period the interest rate is $i = \frac{r}{n}$ and there are $n \cdot t$ time periods in t years.

So, compound interest is

Compound interest is calculated by the formula $A = P\left(1 + \frac{r}{n}\right)^{nt}$

where

A = amount after t years
P = Prinicipal
r = interest rate per year
n = number of times compounded per year
t = number of years

8. Find the value of amount of an investment of \$$10,000$ invested for 5 years at 6% compounded quarterly.

9. Find the present value of \$$10,000$ invested at a rate of 8% compounded monthly for 5 years.

What happens as the number of compounding periods increase infinitely?

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

Continuously compounded interest is calculated by the formula $A = Pe^{rt}$ where

10. Find the amount if $\$1,000$ were invested at 11% compounded continuously for 1 year.

Population growth can be modeled with the exponential formula:

$$P(t) = P_0 e^{rt}$$

where P_0 is the initial population, r is the growth rate (as a decimal), t is time, and $P(t)$ is the population at any time t.

This formula is similar to the continuously compounding formula.

11. The population of a city was $250,000$ and is growing at a relative rate of about 5% a year. What would you expect the population to be in 10 years?

A model which more accurately describes a population growth is a logistic growth model of the form $f(t) = \dfrac{a}{b + ce^{-kt}}$.

12. The population of a certain species of bird is limited by the type of habitat required for nesting. The population is modeled by the logistic growth formula $f(t) = \dfrac{5600}{.5 + 27.5e^{-.044t}}$.

a) What is the initial population?

b) What is the limiting value?

c) Find the graph.