More with Matrices and the Calculator Video Lecture

Sections 11.1 and 11.2

Course Learning Objectives:

1) Solve systems of linear equations using technology.
2) Perform the algebra of matrices, find inverses of matrices, and solve matrix equations.

Weekly Learning Objectives:

1) Solve systems of equations by elimination.
2) Identify inconsistent systems of equations containing two variables.
3) Express the solution of a system of dependent equations containing two variables.
4) Solve systems of three equations containing three variables.
5) Identify inconsistent systems of equations containing three variables.
6) Express the solution of a system of dependent equations containing three variables.
7) Perform row operations on a matrix.
8) Solve a system of linear equations using matrices.

More Matrices and the Calculator

An equation of the form $Ax + By = C$ is called LINEAR since its graph is a line. For a system of linear equations , exactly one of the following is true.

1. The system has exactly one solution. Such a system is said to be

2. The system has no solutions. Such a system is said to be

3. The system has infinitely many solutions. Such a system is said to be

This is easy to see geometrically if we have a system in two variables. Although it is more difficult to visualize if we have a system of n equations with n variables, the statement is still true.

In solving a system of equations we are attempting to find a solution by creating an equivalent system whose solutions are more obvious for us to see. An equivalent system is a system whose solutions are the same as the original system. If we use the elimination method, then there are 3 operations that can be used to create an equivalent system.

1. Add a nonzero multiple of one equation to another.

2. Multiply an equation by a nonzero constant.

3. Interchange the position of two equations.

Solve each of the systems:

1.
$$\begin{array}{rcrcr} x & - & 3y & = & -8 \\ x & + & 4y & = & 13 \end{array}$$

2.
$$\begin{array}{rcrcr} x & - & 3y & = & -8 \\ 2x & - & 6y & = & 12 \end{array}$$

3.
$$\begin{array}{rcrcr} x & - & 3y & = & 8 \\ 2x & - & 6y & = & 16 \end{array}$$

Note: A system is said to be in triangular form if the second equation does not contain the first variable, the third equation does not contain either the first or second variable, etc. The following is an example of a system in a triangular form.

$$\begin{aligned} x - 3y + 3z &= 8 \\ y - 5z &= 16 \\ z &= 4 \end{aligned}$$

It is easy to solve a system that is in triangular form by using back substitution. Our goal in this section is to change a system of linear equations into an equivalent system that is in triangular form.

This process is referred to as Gaussian elimination.

We can express a system of equations as a rectangular array of numbers called a MATRIX. Matrices (plural of matrix) provide us will an efficient method for solving a system of linear equations.

A $m \times n$ matrix is a rectangular array of m rows and n columns.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & . & . & . & a_{1n} \\ a_{21} & a_{22} & a_{23} & . & . & . & a_{2n} \\ a_{31} & a_{32} & a_{33} & . & . & . & a_{3n} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ a_{m1} & a_{m2} & a_{m3} & . & . & . & a_{mn} \end{bmatrix}$$

The numbers a_{ij} are the entries of the matrix. The subscript on the a_{ij} entry indicates that it is the i^{th} row and the j^{th} column.

We write a system of linear equations in matrix form by writing only the coefficients and constants that appear in the equations. The resulting matrix is called the augmented matrix for the system.

$$\begin{array}{ccccccc} x & - & 3y & + & 3z & = & 4 \\ 2x & + & y & - & 5z & = & -11 \\ x & - & 2y & + & z & = & 0 \end{array} \qquad \begin{bmatrix} 1 & -3 & 3 & 4 \\ 2 & 1 & -5 & -11 \\ 1 & -2 & 1 & 0 \end{bmatrix}$$

We can perform the operations referred to in solving a system of equations on the matrix form of the equation. When performed on the matrix form of an equation they are referred to as Elementary Row Operations and expressed in the following way.

1. Add a nonzero multiple of one row to another.

2. Multiply a row by a nonzero constant.

3. Interchange two rows.

To solve a system of linear equations using its augmented matrix we use the elementary row operations to arrive at a matrix that is "triangular". The matrix is said to be in Row Echelon Form (REF). This method is often referred to as Gaussian Elimination.

Row Echelon Form:

1. The first nonzero number in each row (reading from left to right) is 1. This is called the **leading entry**.

2. The leading entry in each row is to the right of the leading entry in the row immediately above it.

3. All rows consisting entirely of zeros are at the bottom of the matrix.

The following matrix is in row echelon form. $\begin{bmatrix} \mathbf{1} & 2 & 3 & -3 \\ 0 & \mathbf{1} & 4 & 2 \\ 0 & 0 & \mathbf{1} & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Reduced Row-Echelon Form (RREF):

Also satisfies the following condition:

4. Every number above and below each leading entry is a 0.

The following matrix is in reduced row-echelon form.

$$\begin{bmatrix} \mathbf{1} & 0 & 0 & 2 \\ 0 & \mathbf{1} & 0 & -1 \\ 0 & 0 & \mathbf{1} & 5 \end{bmatrix}$$

This process is also referred to as Gauss-Jordan elimination.

Solve using a matrix:

$$\begin{array}{rcrcrcr} 4x & + & 8y & - & 4z & = & 4 \\ 3x & + & 6y & + & 5z & = & -13 \\ -2x & + & y & + & 12z & = & -17 \end{array}$$

Find REF and RREF

Now solve using your calculator:

A system which has no solution (system is said to be inconsistent) has a reduced row echelon form similar to the following.

rref: $\begin{pmatrix} 1 & -3 & 2 & 12 \\ 0 & 1 & 1 & -10 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

A system which has an infinite number of solutions (system said to be dependent) has a reduced row echelon form similar to either of the following.

rref: $\begin{pmatrix} 1 & 0 & 0 & -4 & 0 \\ 0 & 1 & 0 & 0 & 5 \\ 0 & 0 & 1 & 3 & 0 \end{pmatrix}$ Assume variables are $x,\ y,\ z,\ w$

The solution to the system corresponding to the augmented matrix above is represented as

rref: $\begin{pmatrix} 1 & 0 & -3 & -4 & 0 \\ 0 & 1 & 0 & 0 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ Assume variables are $x,\ y,\ z,\ w$

The solution to the system corresponding to the augmented matrix above is represented as

5.

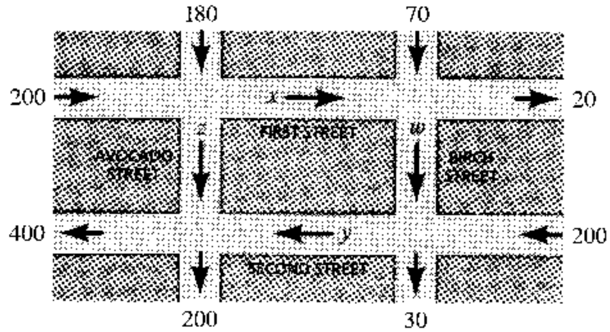
$$
\begin{aligned}
6x - y - z &= 4 \\
-12x + 2y + 2z &= -8 \\
5x + y - z &= 3
\end{aligned}
$$

6.

$$\begin{aligned} x - 2y + z &= 0 \\ 2x + 2y - 3z &= -3 \\ y - z &= -1 \\ -x + 4y + 2z &= 13 \end{aligned}$$

7. Carletta has $10,000 to invest. As her financial consultant, you recommend that she invest in Treasury bills that yield 6%, Treasury bonds that yield 7%, and corporate bonds that yield 8%. Careltta wants to have an annual income of $680, and the amount invested in the corporate bonds must be half that invested in the Treasury bills. Find the amount in each investment.

8. The following diagram shows traffic flow in a section of the city.



The arrows indicate one-way streets. The numbers on the diagram show how many cars enter or leave this section of the city via the indicated street in a certain one hour period. The variables x, y, z, and w represent the number of cars that travel along the portions of First, Second, Avocado, and Birch streets during the period. Find x, y, z and w, assuming that none of the cars involved in this problem stop or park on any of the streets shown in the picture.