Rational Functions Video Lecture

Sections 4.4 and 4.5

Course Learning Objectives:

1) Demonstrate an understanding of functional attributes such as domain and range. Determine these attributes for a function given its graph and/or equation.
2) Graph rational functions and use such graphs to solve applied problems and to understand the significance of attributes of the graph to such applied problems.
3) Identify and articulate the significance of graphical components such as x-intercepts and horizontal asymptotes in a mathematical model/application.

Weekly Learning Objectives:

1) Find the domain of a rational function.
2) Find the vertical asymptotes of a rational function.
3) Find the horizontal or oblique asymptotes of a rational function.
4) Find the x- and y-intercepts of rational functions.
5) Graph rational functions.
6) Analyze the graph of a rational function.
7) Solve applied problems involving rational functions.

Rational Functions

A rational number is any number of the form $\frac{p}{q}$ where p and q are integers with q not equal to 0.

A rational function is the quotient of two polynomial functions:

$$\frac{P(x)}{Q(x)}$$

The domain of a rational function is all real numbers except for when:

$$Q(x) = 0$$

A rational function is in lowest terms if P(x) and Q(x) have no common factors and a rational function is proper if the degree of P(x) is less than the degree of Q(x).

Think about the graph of f(x) = $\frac{1}{x}$

Domain:

Notice:

$f(x) \rightarrow$ as $x \rightarrow 0^+$

$f(x) \rightarrow$ as $x \rightarrow 0^-$

We say that x = 0 is a vertical asymptote.

Notice: $f(x) \rightarrow$ as $x \rightarrow +\infty$

$f(x) \rightarrow$ as $x \rightarrow -\infty$

We say that y = 0 is a horizontal asymptote.

The line x = a is a vertical asymptote of the function y = f(x) if any of the following are true:

$f(x) \rightarrow \pm\infty$ as $x \rightarrow a^+$ or $x \rightarrow a^-$

The line y = b is a horizontal asymptote of the function y = f(x) if:

$f(x) \rightarrow b$ as $x \rightarrow +\infty$ or $x \rightarrow -\infty$

Asymptotes are not part of the graph but they serve as an aid in sketching the graph and provide information about the way the graph looks. A horizontal asymptote is a special type of tail approximation. Sometimes the tails of a function will align with a slanted line yielding a "slant" or "oblique" asymptote.

How to find asymptotes:

To find a vertical asymptote for the rational function $\frac{P(x)}{Q(x)}$ **find the values of c that make Q(c) = 0, but not P(c) = 0.**

1. **Factor P(x) and Q(x).**
2. **If any factors of P(x) and Q(x) match, then the solutions of those factors will result in HOLES in the graph.**
3. **If any factors of Q(x) do not match factors of P(x), then the solutions to those factors will give vertical asymptotes.**

Find the vertical asymptotes and domain of the following functions:

$$f(x) = \frac{x}{x^2-4}$$

$$f(x) = \frac{x-3}{x+4}$$

$$f(x) = \frac{3x}{x^2+4}$$

$$f(x) = x^2+5x+6$$

$$f(x) = \frac{x-1}{x^3-3x^2-4x+12}$$

$$f(x) = \frac{x-3}{x^2-9}$$

$$f(x) = \frac{x^2-x-2}{x^2-3x-4}$$

To find horizontal asymptotes analyze the numerator and denominator separately and then reduce. Remember that the highest degree term determines the tail behavior of a function. We will always get a result of the form

i) ax^n (No horizontal asymptote)

ii) c (y = c is the equation of the horizontal asymptote) or

iii) $\frac{a}{x^n}$ (y = 0 is the equation of the horizontal asymptote).

If the degree of the numerator is one more than the degree of the denominator, the function will have a linear oblique asymptote. If the degree of the numerator is two more than the degree of the denominator, the function will have a parabolic oblique asymptote.

To find the equation of the oblique asymptote rewrite the function as a quotient with a remainder (divide the denominator into the numerator). The equation of the oblique asymptote will be y = Quotient.

Find the equation of the horizontal or oblique asymptote:

$$f(x)=\frac{2x+5}{3-x}$$

$$f(x)=\frac{x^2+1}{x^3-5x+2}$$

$$f(x)=\frac{x^3+3x^2-2}{x^2+1}$$

$$f(x)=\frac{x^4-5x^3+2}{x^2-3x}$$

Guidelines for graphing a rational function

1. Factor the numerator and denominator. The zeros of the numerator are the x-intercepts of the graph. The zeros of the denominator lead to the vertical asymptotes of the function. (Assuming the zeros of the numerator are different from the zeros of the denominator, otherwise there is a hole at the zero.) If the zero (for either the numerator or denominator) has even multiplicity, the graph will not change sign in the vicinity of the intercept (asymptote) and if the zero (for either the numerator or denominator) has odd multiplicity, the graph will change sign in the vicinity of the intercept (asymptote).

2. Find the y-intercept by finding the value of the function when x=0.

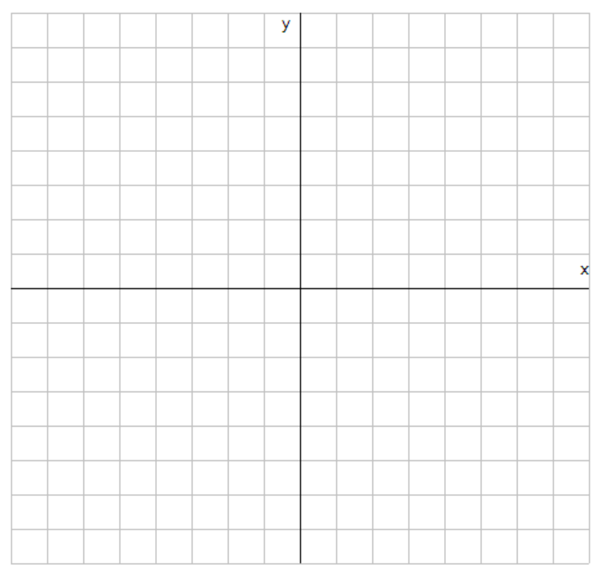
3. Plot the x and y intercepts and note the multiplicity of the intercepts. Sketch the vertical asymptotes (noting multiplicity).

4. Determine the tail behavior - may lead to horizontal or oblique asymptotes.

5. Sketch a smooth curve that passes through the required points and has the appropriate end behavior.

Sketch a graph for each of the following:

$$f(x) = \frac{(2x+3)(x-1)}{(x+2)(x-4)}$$



x - intercept:

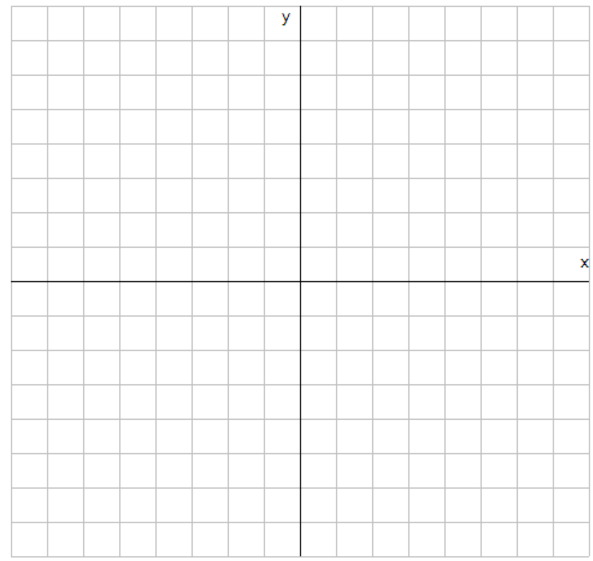
y - intercept:

Horizontal Asymptote/Oblique Asymptote:

Vertical Asymptote:

$$f(x)=\frac{(x+1)}{(x+2)(x-4)}$$

x - intercept:



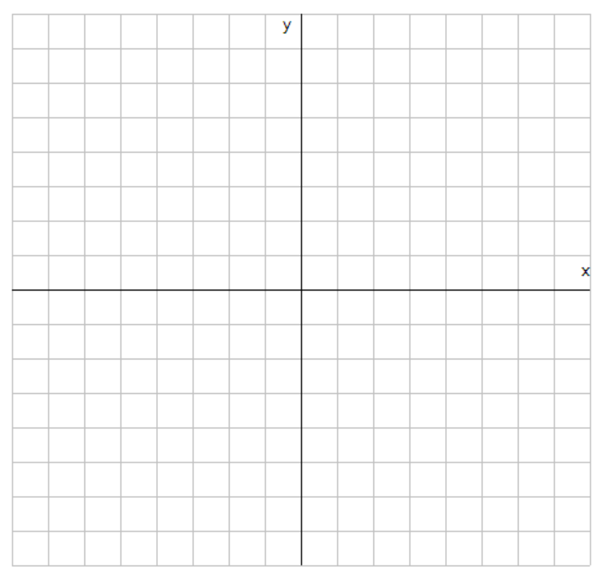
y - intercept:

Horizontal Asymptote/Oblique Asymptote:

Vertical Asymptote:

$$f(x)=\frac{(x-1)(x-2)}{(x+3)}$$

x - intercept:



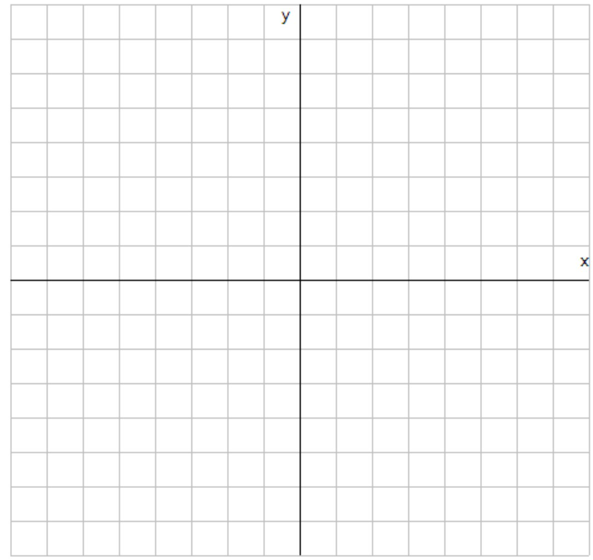
y - intercept:

Horizontal Asymptote/Oblique Asymptote:

Vertical Asymptote:

$$f(x) = \frac{(x+1)^2}{(x+3)^2}$$

x - intercept:



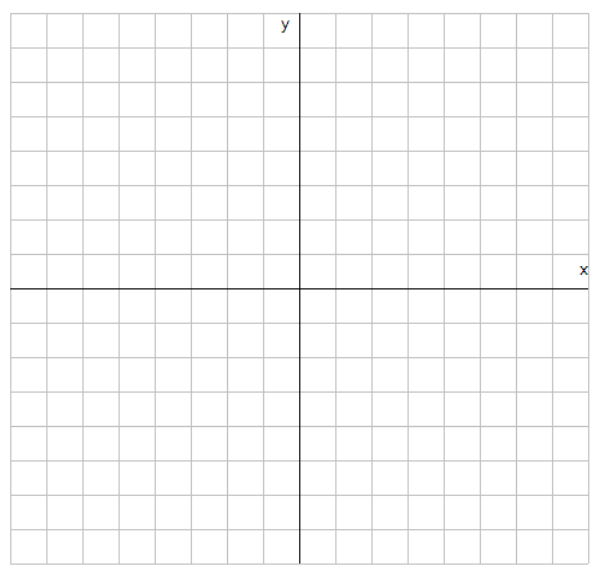
y - intercept:

Horizontal Asymptote/Oblique Asymptote:

Vertical Asymptote:

$$f(x) = \frac{(x-5)(x+3)(1-x)}{(x+4)(x+2)^2}$$

x - intercept:

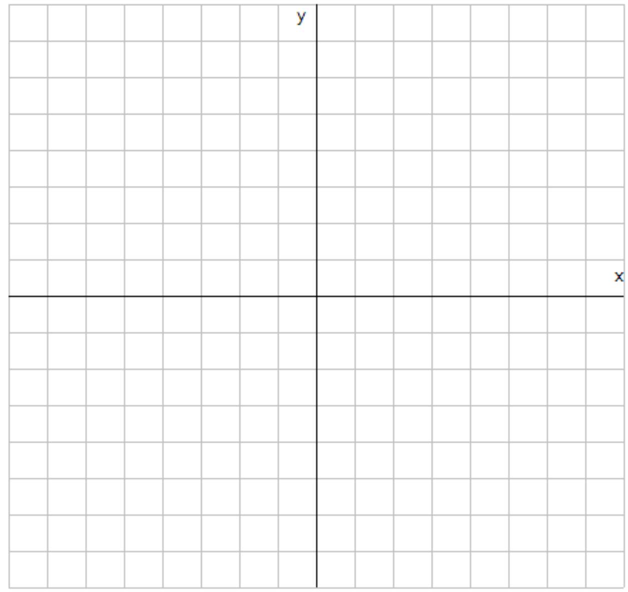


y - intercept:

Horizontal Asymptote/Oblique Asymptote:

Vertical Asymptote:

$$f(x) = \frac{x^2 - x - 6}{x^2 - 5x + 6}$$



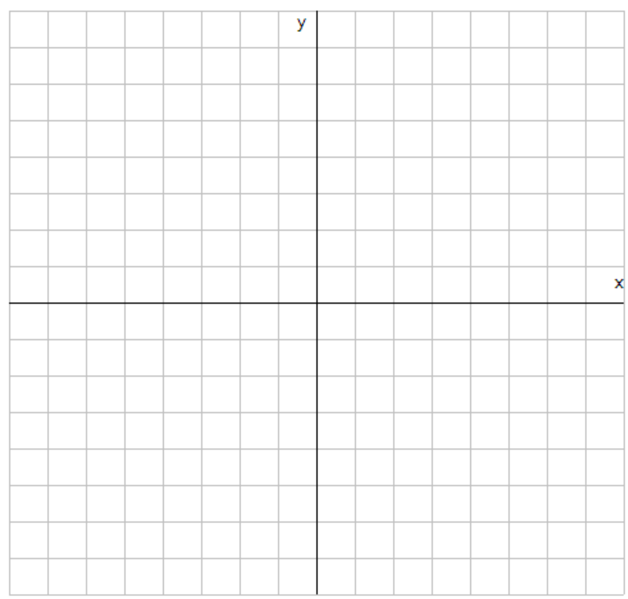
x - intercept:

y - intercept:

Horizontal Asymptote/Oblique Asymptote:

Vertical Asymptote:

$$f(x) = \frac{(x^2 - 2x - 3)(x - 5)}{(x - 2)(x^2 - 4x + 3)}$$



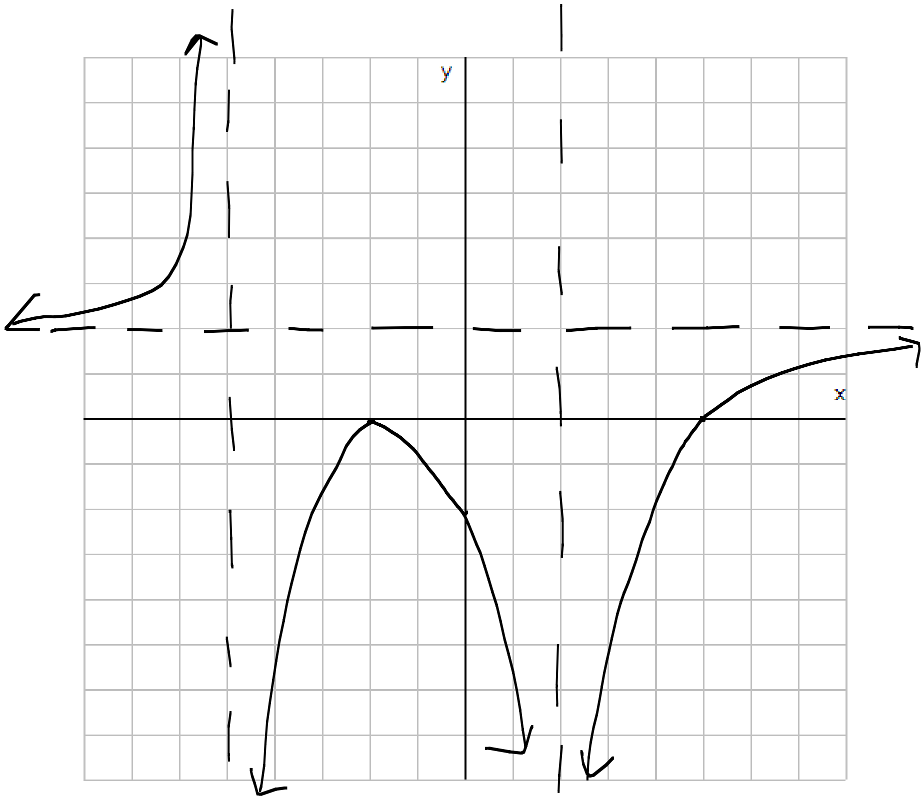
x - intercept:

y - intercept:

Horizontal Asymptote/Oblique Asymptote:

Vertical Asymptote:

Find the equation of a rational function that would fit the graph below:



An application of rational functions:

Find the dimensions of a cylinder of minimum cost if the cylinder is to have a capacity of 500 cubic centimeters and the cost factors are as follows. The top and bottom of the cylinder cost 5 ¢/cm^2 **and the lateral surface costs 2** ¢/cm^2.