The Complex Plane and De Moivre's Theorem Video Lecture

Section 9.3

Course Learning Objectives:
Graph polar equations with and without technology.

Weekly Learning Objectives:

1) Plot points in the complex plane.
2) Convert a complex number from rectangular form to polar form.
3) Find products and quotients of complex numbers in polar form.
4) Use De Moivre’s theorem.
5) Find complex roots.

The Complex Plane and De Moivre's Theorem

Recall that a complex number is of the form $x + yi$ where x and y are any real numbers.

If we let $z = x + yi$, then we can show this geometrically as the point (x, y) in the xy-plane.

Using this model, each point in the plane corresponds to a complex number and every complex number corresponds to a point in the plane. We call this the **complex plane**. The $x-$ axis is the **real axis**, because any point that lies on the real axis is of the form $z = x + 0i = x$, a real number. The $y-$ axis is called the **imaginary axis**, because any point that lies on it is in the form $z = 0 + yi = yi$, a pure imaginary number.

Plot the point $z = 2 - i$ in the complex plane.

Definition: Let $z = x + yi$ be a complex number. The **magnitude** or **modulus** of z, denoted by $|z|$, is defined as the distance from the origin to the point (x, y). That is,

$$|z| = \sqrt{x^2 + y^2}$$

The magnitude of z is sometimes called the **absolute value** of z.

Recall that if $z = x + yi$, then its conjugate, denoted by $\overline{z}$, is $\overline{z} = x - yi$.

Notice that $z\overline{z} =$

Therefore, $|z| = \sqrt{z\overline{z}}$

When a complex number is written in the standard form $z = x + yi$, we say it is in **rectangular**, or **Cartesian form**, because (x, y) are the rectangular coordinates of the corresponding point in the complex plane. Suppose that (r, θ) are the polar coordinates of this point, then

$$x = r\text{cos}\theta \qquad y = r\text{sin}\theta$$

Then we can write the complex number $z = x + yi$ in **polar form** as:

$$\boldsymbol{z} = x + yi \ = (r\,cos\theta) \ + \ (r\,sin\theta)i \ = \ \boldsymbol{r(cos\theta + i\,sin\theta)}$$

The angle θ, with $\mathbf{0 \leq \boldsymbol{\theta} < 2\boldsymbol{\pi}}$, is called the **argument of $\boldsymbol{z}$**.
The magnitude of z is $|z| = r$.

Write the expression for $z = \sqrt{3} - i$ in polar form.

Plot the point corresponding to $z = 2(cos\,30° + \ i\,sin 30°)$ in the complex plane, and write an expression for z in rectangular form.

The polar form of a complex number provides an alternative method for finding products and quotients of complex numbers.

Theorem: Let $z_1 = r_1(cos\,\theta_1 + i\,sin\theta_1)$ and $z_2 = r_2(cos\,\theta_2 + i\,sin\theta_2)$ then,

$$\boldsymbol{z_1 z_2 = r_1 r_2[cos(\theta_1 + \theta_2)\ +\ i\,sin(\theta_1 + \theta_2)]}$$

$$\boldsymbol{\tfrac{z_1}{z_2} = \tfrac{r_1}{r_2}[cos(\theta_1 - \theta_2)\ +\ i\,sin(\theta_1 - \theta_2)]}$$

Proof:

If $z = 3(cos\,20° \ + i\,sin20°)$ and $w = 5(cos\,100° \ +\ i\,sin100°),$ find the following (leave your answer in polar form):

a) zw

b) $\frac{z}{w}$

Let $z = r(cos\,\theta\ + i\,sin\,\theta)$, then find:

$z^2\ =$

$z^3\ =$

$z^4 =$

De Moivre's Theorem: If $z = r(cos\,\theta\ + i\,sin\,\theta)$ is a complex number, then

$$z^n = r^n(cos\,n\theta\ + i\,sin\,n\theta)$$

where $n \geq 1$ is a positive integer.

Find $[2(cos\,20^\circ\ + i\,sin 20^\circ)]^3$ in standard complex form.

Write $(1+i)^5$ in standard form.

Now we will focus on finding roots of complex numbers. If w is a complex number, then the solutions to $z^2 = w$ would give the two complex square roots of w. The solutions to $z^3 = w$ would give the three complex cube roots of w, etc.

If $w = r(\cos\theta_0 + i\,\sin\theta_0)$ is a complex number and n is an integer with $n \geq 2$, then there will be n distinct complex roots of w given by the formula:

$$z_k = \sqrt[n]{r}\,[\cos\left(\frac{\theta_0}{n} + \frac{2k\pi}{n}\right) + i\,\sin\left(\frac{\theta_0}{n} + \frac{2k\pi}{n}\right)]$$

where $k = 0, 1, 2, 3, ..., n - 1.$

Find the complex cube roots of $-1 + \sqrt{3}\,i$. Leave your answer in polar form, with the argument in degrees.